MATH PLUS

Practical Life Uses and Solutions

7

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Preface

Math PLUS (Practical Life Uses and Solutions) 7 is part of a junior high school math textbook series that is designed to cater to the needs of 21st-century learners. Each book progressively guides the students in mastering mathematics concepts and prepares them for higher-level mathematics learning.

Each chapter begins with a real-life illustration of the topics involved. Each lesson in a chapter also starts with a real-life illustration of the lesson topic that is related to the chapter's theme. The chapter and lesson openers motivate the students to learn about the importance of mathematics in their lives.

Each lesson is composed of the following components:

Probe and Learn provides discussions of the topics using illustrations, examples with step-by-step solutions, and key concepts. This component has the following subcomponents:

Key Concept highlights important terms, laws, formulas, properties, postulates, or theorems that the students should look into to better understand the lesson.

Do You Know gives interesting information or trivia about a specific term or concept mentioned in the lesson.

Practice provides assessment items in varying levels of difficulty to allow the students to master and apply what they have learned. This component is divided into three parts:

Concepts and Skills is composed of knowledge-based items about the topics discussed.

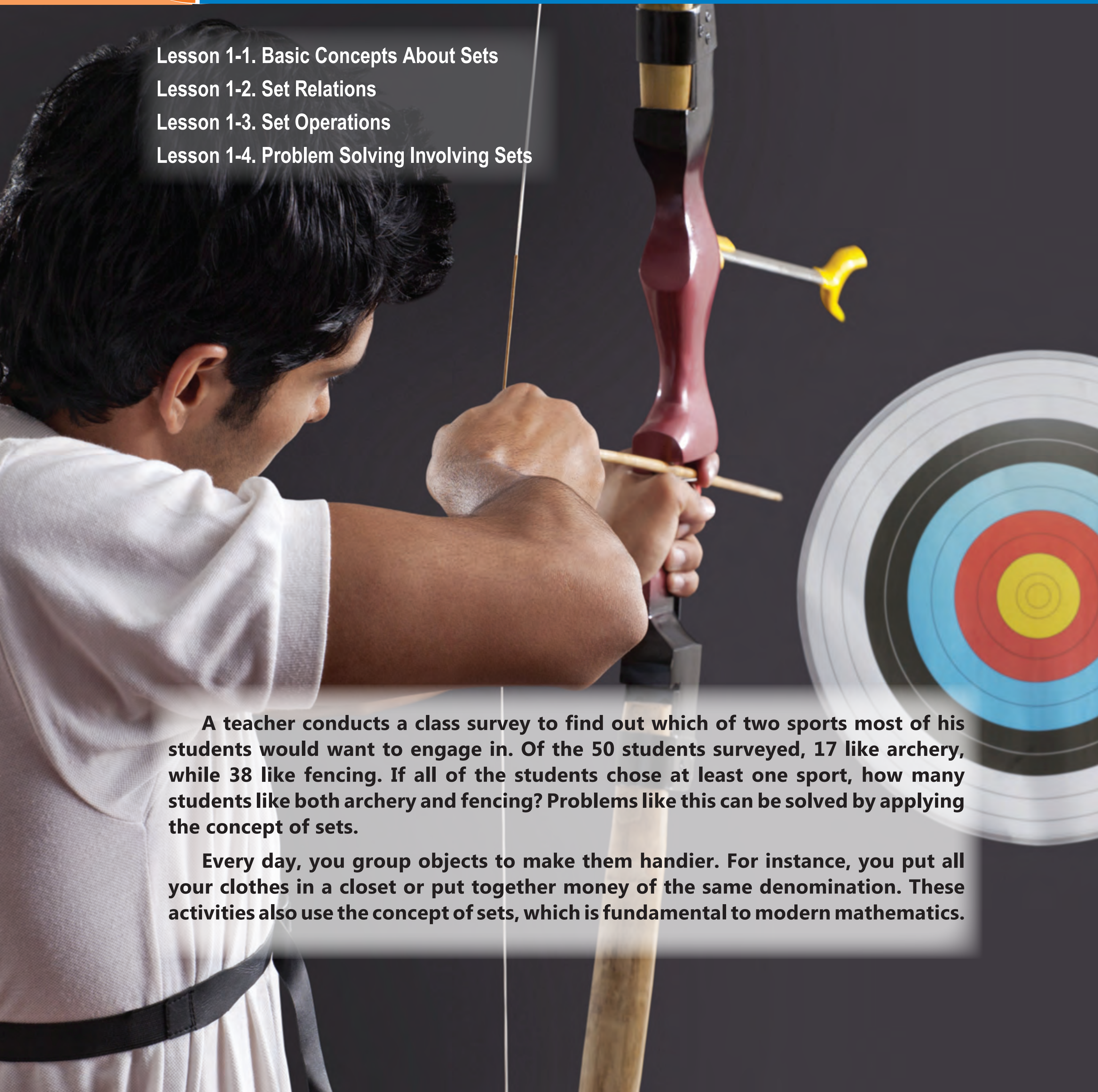
Applications is composed of problem-solving items that focus on real-life applications of the topics.

Enrichment Exercises is composed of items that go beyond the topics discussed to enhance the students' critical thinking skills.

Each chapter concludes with a **Chapter Assessment** to measure the acquired knowledge and skills of the students.

CHAPTER 1

Introduction to Sets



A teacher conducts a class survey to find out which of two sports most of his students would want to engage in. Of the 50 students surveyed, 17 like archery, while 38 like fencing. If all of the students chose at least one sport, how many students like both archery and fencing? Problems like this can be solved by applying the concept of sets.

Every day, you group objects to make them handier. For instance, you put all your clothes in a closet or put together money of the same denomination. These activities also use the concept of sets, which is fundamental to modern mathematics.

LESSON 1-1

Basic Concepts About Sets

Learning Objectives

At the end of the lesson, you should be able to:

- define sets
- describe sets using the roster method and the rule method
- identify the elements of a set
- find the cardinality of a set
- classify sets according to size

PROBE AND LEARN

Look at the following list of sports:

archery	gymnastics	swimming
athletics	handball	table tennis
badminton	hockey	taekwondo
basketball	judo	tennis
boxing	pentathlon	triathlon
cycling	rowing	volleyball
equestrian	sailing	water polo
fencing	shooting	weightlifting
football	softball	wrestling

Have you heard or played any of these sports? What do they have in common?

These are the sports played in the Summer Olympic Games. You might have played some of them or watched them being played. Look again at the list of sports. Try to group the sports based on the following categories:

Water Sports	Ball Games
Exciting Games	**Popular Sports**

Now compare your list with that of your classmate. Did you write the same sports under each category? In which categories did you list different sports? Why did you have different lists in those categories?

Definition of a Set

In the previous activity, the games you listed under each category form a group. Take note, however, that not all groups are sets. Consider the sports that you categorized as water sports. If you are familiar with all the sports in the given list, you could identify the water sports as follows:

rowing, sailing, swimming, and water polo.

This group of sports is an example of a set. It is a set because the common characteristic of the identified sports is clear and definite. All of them are played in the water. On the other hand, the sports that are listed as "exciting games" may vary. What is exciting for you may not be exciting for others. Hence, the group of sports under "exciting games" is not a set. Study the following definition of a set:

Key Concept

A **set** is a well-defined collection of objects. Such objects are called the **elements** of the set.

Remember that the elements of a set should be described in a way that each element that belongs to the set can be easily identified. This may be done by giving a clear rule that every element of the set satisfies. Study the examples.

Example 1

Tell whether or not each group is a set. Explain.

a. group of varsity players in your school

b. group of students who participated in a basketball competition

c. group of good dancers in your class

d. group of popular volleyball players

Answers:

a. It is a set. Your school has a definite roster of varsity players.

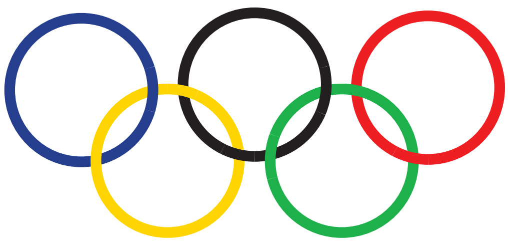
b. It is a set. You can identify the students who participated in the competition.

c. It is not a set. People may have different standards in recognizing good dancers.

d. It is not a set. Popularity is based on opinions, which may vary from one person to another.

Naming a Set and Its Elements

A set is conventionally denoted by a capital letter. You may use a letter that can remind you of the common characteristic or rule that describes the elements the set. For example, you could name the set of colors of the rings in the Olympic symbol as set O. The elements of set O are blue, yellow, black, green, and red.



The Olympic rings symbol was designed by Pierre de Coubertin in 1913 for the Paris Congress of the Olympic Movement. It is made up of five interlocking rings—blue, yellow, black, green, and red—representing the five major regions of the world.

Set membership is denoted by the symbol $\in$, which is the lowercase Greek letter epsilon. Suppose that set B is the set of countries whose representatives won in at least one of the QubicaAMF Bowling World Cup (BWC) tournaments from 1965 to 2018. The Philippines belongs to that set of countries. Hence, you can write:

$$\text{Philippines} \in B,$$

which is read as "the Philippines is an element of set B."

To denote that an element is not a member of a set, the symbol $\notin$ is used. For instance, Indonesia did not win in any BWC tournament from 1965 to 2018. So you can write:

$$\text{Indonesia} \notin B,$$

which is read as "Indonesia is not an element of set B."

Example 2

Suppose that P is the set of Philippine Basketball Association (PBA) players in 2017. Which of the following is true?

a. Tim Yap $\notin P$

b. Tim Cone $\in P$

c. Allan Caidic $\notin P$

d. Jordan Clarkson $\in P$

Answers:

a. True; Tim Yap is not a PBA player.

b. False; Tim Cone is a PBA coach, not a player. Thus, Tim Cone $\notin P$.

c. True; Allan Caidic is a PBA player who played during the 1990s, not in 2017.

d. False; Jordan Clarkson is an NBA player, not a PBA player. Hence, Jordan Clarkson $\notin P$.

Methods to Describe Sets

The following are the two commonly used methods to describe sets: (1) the *roster method* or *listing method*; and (2) the *rule method* or *set-builder notation*.

Key Concept

In the **roster method** or **listing method** of describing a set, the elements of the set are listed between braces. The elements in the list are separated by commas.

For example, suppose that set V is the set that contains the vowels of the English alphabet. Using the roster method, you can describe this set as follows:

$$V = \{a, e, i, o, u\}.$$

You can see that the elements of V are the letters a, e, i, o, and u. Since there are only five elements in V, it is easy to enumerate them all. Note that when writing the elements of a set, the order of the elements does not matter. So you can also write set V as $\{o, u, a, e, i\}$ or $\{e, i, o, u, a\}$. Also, remember that the elements in a set should be distinct; that is, the elements should not be repeated in the list.

In cases when a set has many elements, an ellipsis (...) is used to indicate that the list of elements continues up to the last given element. For instance, suppose that set C is the set of the first 1,000 counting numbers. You may describe set C as follows:

$$C = \{1, 2, 3, ..., 1{,}000\}.$$

Key Concept

In the **set-builder notation** or the **rule method** of describing a set, the property or properties that each element of the set satisfies are stated.

When using the set-builder notation, the rule should be written clearly so as to include only those objects that satisfy the described property. In set-builder notation, the general form a set can be written as:

$$A = \{x \mid x \text{ is } P\},$$

where P is the defining property. This statement is read as "set A is the set of all x's such that x satisfies P."

Consider again the two sets, C and V, that you previously described using the roster method. In set-builder notation, you can write the sets as follows:

$V = \{y \mid y \text{ is a vowel in the English alphabet}\}$; and

$C = \{z \mid z \text{ is a counting number from 1 to 1,000}\}$.

You may use either the roster method or the rule method, whichever is more practical, to describe a given set.

EXAMPLE 3

Describe each set using the rule method.

a. $W = \{0, 1, 2, 3, \ldots\}$

b. $B = \{11, 12, 13, 14, 15, 16, 17, 18, 19\}$

c. D = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}

d. C = {monitor, central processing unit, keyboard, mouse, speaker, printer, scanner}

Answers:

a. $W = \{x \mid x \text{ is a whole number}\}$

b. $B = \{y \mid y \text{ is a counting number greater than 10 but less than 20}\}$

c. $D = \{z \mid z \text{ is a day of the week}\}$

d. $C = \{v \mid v \text{ is a part of a computer system}\}$

Kinds of Sets

Sets may be classified based on the number of their elements. Study the following definition:

Key Concept

The **cardinality** of a set refers to the number of elements of the set. In symbols, the cardinality of a set A is denoted as $n(A)$.

Consider the following set:

$$P = \{x \mid x \text{ is a planet in the solar system}\}.$$

Using the roster method, set P may be described as follows:

P = {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune}.

You can say that the cardinality of set P is 8. In symbols, $n(P) = 8$.

Now consider the following set:

$A = \{x \mid x$ is a planet in the solar system whose name starts with the letter A$\}$.

Is there a planet whose name starts with the letter A? Since there is no such planet, then A has no elements. Thus, $n(A) = 0$.

Key Concept

A set that has no elements is called an **empty set** or a **null set**. It is denoted as ∅ or { }.

Since $n(A) = 0$, you can say that A is an empty set.

Key Concept

If the cardinality or number of elements of a set is a counting number, then it is called a **finite set**. A set is defined as a finite set if the cardinality of the set is a fixed or known counting number n. Otherwise, the set is not finite and is called an **infinite set**.

The elements of a finite set are limited; thus, such elements can be completely listed down. For example, $V = \{a, e, i, o, u\}$ is a finite set since $n(V) = 5$. Also, although an empty set has no element (or zero element), and 0 is not a counting number, an empty set is also considered as a finite set. An example of an infinite set is the set of counting numbers, $C = \{1, 2, 3, \ldots\}$. It is an infinite set because there are infinitely many counting numbers.

Is the set of the stars in the sky finite or infinite? You may think that since there are so many stars, then it is impossible to count all of them. However, it has been known that there are approximately 200,000,000 stars in the sky. Although this set is very large, it is considered finite since its cardinality is a whole number.

PRACTICE 1-1

Concepts and Skills

Write *True* if the statement is true; otherwise, write *False*.

1. The empty set $\varnothing$ is the same as {0}.
2. The set of fractions is an infinite set.
3. The set of proper fractions is a finite set.
4. The size of an empty set *cannot* be described.
5. The set {1, 2, 3, 4, 5} is the same as the set {5, 4, 3, 2, 1}.
6. The elements of a set may be written more than once in a set.
7. It is impossible to enumerate all the elements of an infinite set.
8. Any set could be described using both the roster method and the rule method.
9. In the rule method, the elements of a set are listed.
10. In the roster method, a set is described by specifying the properties its elements should satisfy.

Tell if each group is a set or not. Explain.

11. collection of large numbers
12. group of sea mammals
13. group of distinct letters in your first name
14. list of names of girls with long hair in your class
15. collection of fiction books in your school library

Describe each set using the roster method.

16. set *P* containing perfect square numbers
17. set *F* containing the colors in the Philippine flag
18. set *L* containing the leap years in the 21st century
19. set *T* containing three-digit palindromes
20. set *O* containing the siblings of an only child in a family

Describe each set using the rule method.

21. $U = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$
22. $A = \{2, 3, 5, 7, 11, 13, 17, 19, \ldots, 97\}$
23. *M* = {April, June, September, November}
24. $C = \{5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 49\}$

25. P = {Aguinaldo, Quezon, Laurel, Osmeña, Roxas, Quirino, Magsaysay, Garcia, Macapagal, Marcos, Aquino, Ramos, Estrada, Arroyo, Aquino, Duterte}

For items **26** to **30**, find the cardinality of the sets described in **16** to **20**.

26. $n(P)$

27. $n(F)$

28. $n(L)$

29. $n(T)$

30. $n(O)$

For items **31** to **35**, classify the sets described in **21** to **25** as finite or infinite.

31. U

32. A

33. M

34. C

35. P

For items **36** to **40**, suppose that M is the set of the sums of any two consecutive whole numbers between 10 and 50.

36. Describe set M using the roster method.

37. Describe set M using the rule method.

38. Classify M as finite or infinite.

39. Find the cardinality of M.

40. Tell whether or not $21 \in M$. Explain.

Applications

Answer each item completely.

41. Suppose that you will make a model of the solar system. Name all the objects that will be represented in your model. Describe the objects in your model using the roster method.

42. Condiments are food seasonings that improve the flavor of food. You want to enhance the flavor of the food you are eating. What set of condiments are you going to choose? Describe it as a set containing five elements using the roster method.

43. Suppose that you will grow some herbal plants in your school backyard. Give a set of ten plants that you would consider planting. Use the roster method.

44. Maria plans to establish a pet shop. Using the rule method, describe a set of animals that Maria could sell in her shop.

45. Suppose that the sports coordinator in your school organizes a sports fest. Your school has no facilities for outdoor games; so only indoor games can be played. Describe a set of games you could suggest to the sports coordinator. Use the roster method.

Enrichment Exercises

Answer each item completely.

46. Explain why an empty set is a finite set.

47. Give an example of an empty set. Use the rule method to describe the set.

48. Give an example of a finite set with a cardinality of 6. Use the rule method to describe the set.

49. Give an example of a set that can be best described using the rule method.

50. Give an example of a set that can be best described using the roster method.

Lesson 1-2

Set Relations

Learning Objectives

At the end of the lesson, you should be able to:

- identify the relation among sets
- illustrates relations among sets using Venn diagrams

Probe and Learn

Could you imagine yourself playing five different sports within a 12-hour period? You might be amazed at how athletes who play the modern pentathlon sustain their stamina. *Modern pentathlon* is an athletic competition that is composed of five different events that test one's endurance and athletic versatility. This includes shooting, fencing, swimming, horseback riding, and running.

Equal Sets and Equivalent Sets

Suppose that you have the following sets:

$P = \{x \mid x$ is an event in modern pentathlon$\}$; and

E = {shooting, fencing, swimming, horseback riding, running}.

Note that all events in E are the sports described in P. Since sets E and P have the same elements, E and P are said to be *equal sets*. In symbols, you can write $E = P$.

Key Concept

Sets A and B are **equal** if they contain exactly the same elements. In symbols, $A = B$.

When two sets, say C and D, are not equal sets, you can write the relation as $C \neq D$.

Look at the following sets:

$E = \{a, b, c, d, e\}$ $\quad$ $F = \{a, e, b, c, d\}$ $\quad$ $G = \{\text{a, b, c, e, f}\}$

You can verify the following relations:

$E = F$ $\quad$ $E \neq G$ $\quad$ $F \neq G$

Another way to relate sets is to consider the number of their elements. Look at the following sets:

$H = \{1, 4, 9, 16, 25, 36\}$ $\quad$ $I = \{x \mid x$ is a whole number less than 6$\}$

Using the roster method, $I = \{0, 1, 2, 3, 4, 5\}$. You can see that $H \neq I$. However, if you will get the cardinality of each set, you will see that $n(H) = n(I)$. In such case, the two sets are said to be *equivalent*.

Key Concept

Sets A and B are **equivalent** if they have the same cardinality. In symbols, $A \sim B$.

Note that there is a one-to-one correspondence between the elements of equivalent sets; that is, each element of one set can be paired with exactly one element of the other set, and vice versa. In the previously given equivalent sets, H and I, you can denote the one-to-one correspondence as $H \leftrightarrow I$. In the following illustration, notice how each element of H can be paired with exactly one element of I.

$$\begin{array}{cccccccc} H = \{ & 1, & 4, & 9, & 16, & 25, & 36\} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ I = \{ & 0, & 1, & 2, & 3, & 4, & 5\} \end{array}$$

Example 1

Consider the following sets:

$L = \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$

$M = \{x \mid x$ is a multiple of 10 from 10 to 100$\}$

$N = \{x \mid x$ is a number from 5 to 50 that is divisible by 5$\}$

Tell whether each relation is true or false. Explain.

a. $L = M$

b. $L \sim M$

c. $L = N$

d. $N \sim L$

e. $M \neq N$

Answers:

To compare the given sets easily, describe first sets M and N using the roster method.

$L = \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$

$M = \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$

$N = \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$

Based on the given sets, you will have the following:

a. True; L and M have exactly the same elements.

b. True; $n(L) = n(M) = 10$.

c. False; L and N have only some elements in common.

d. True; $n(N) = n(L) = 10$.

e. True; M and N do not have exactly the same elements.

Consider three sets A, B, and C. Suppose that $A \sim B$ and $B \sim C$. Does it follow that $A \sim C$? Look at the given sets in the previous example. Note that $L \sim M$ and $M \sim N$. Could you conclude that $L \sim N$? If yes, is this property true for equality of sets? Study the following table that summarizes the properties of equality and equivalence of sets.

Property	Equality	Equivalence
Reflexive Property	A set is equal to itself. $A = A$	A set is equivalent to itself. $A \sim A$
Symmetric Property	The sets can be written interchangeably on any side of the equality. If $A = B$, then $B = A$.	The sets can be written interchangeably on any side of the equivalence. If $B \sim A$, then $A \sim B$.

Property	Equality	Equivalence
Transitive Property	If set *A* is equal to set *B*, and set *B* is also equal to set *C*, then set *A* is equal to set *C*. If $A = B$ and $B = C$, then $A = C$.	If set *A* is equivalent to set *B*, and set *B* is also equivalent to set *C*, then the set *A* is equivalent to set *C*. If $A \sim B$ and $B \sim C$, then $A \sim C$.

Sets and Subsets

Another relationship involving sets is *set inclusion*, which means that a set may be contained in another set. Consider sets *P* and *Q* as follows:

$P = \{x \mid x \text{ is a student in your class}\}$

$Q = \{y \mid y \text{ is a male student in your class}\}$

Based on the description of sets *P* and *Q*, if a student in your class is male, then he is an element of *Q* and, at the same time, of *P*. In such case, *Q* is a *subset* of *P*.

Key Concept

Q is said to be a **subset** of *P* if every element of *Q* is also an element of *P*. In symbols, it is denoted as $Q \subseteq P$. This is read as "*Q* is a subset of *P*" or "*Q* is contained in *P*."

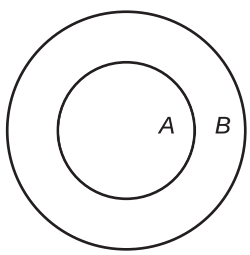
From $Q \subseteq P$, you also say that "*P* is a *superset* of *Q*." It can be written as $P \supseteq Q$. If at least one element of *Q* is not contained in *P*, then *Q* is not a subset of *P*. In symbols, $Q \not\subseteq P$. What can you say about the cardinality of a subset and of its superset?

Set relations can be visualized using a Venn diagram.

Key Concept

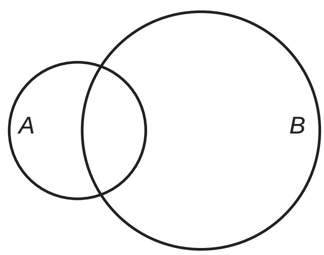
A **Venn diagram** is a visual representation of logical relationships among sets. It uses circles to represent sets. Each circle is labeled with the name of the set it represents.

The relation that set *A* is a subset (or not a subset) of another set *B* can be illustrated in the following Venn diagrams:



$A \subseteq B$

All elements in set *A* are contained in set *B*.



$A \not\subseteq B$

Not all elements in set *A* are contained in set *B*.

Key Concept

If A and B are non-empty sets and A is a subset of B, where $n(A) < n(B)$, then A is said to be a **proper subset** of B. In symbols, it is denoted as $A \subset B$. Note that $A \neq B$.

EXAMPLE 2

Consider the following sets:

$X = \{1, 3, 5, 7, 9\}$

$Y = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

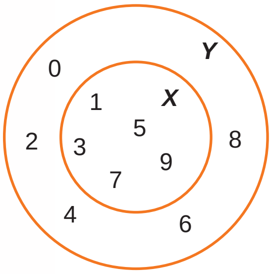
$Z = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$

Illustrate each relation using a Venn diagram.

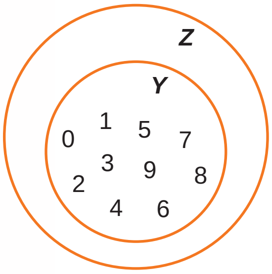
a. $X \subset Y$ b. $Y \subset Z$ c. $X \subset Z$

Answers:

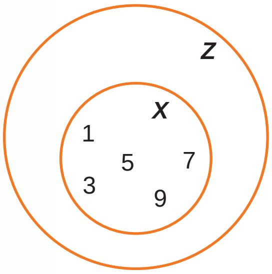
a. Since all the elements of X are found in Y, and $n(X) < n(Y)$, then $X \subset Y$ is illustrated in the diagram on the right.



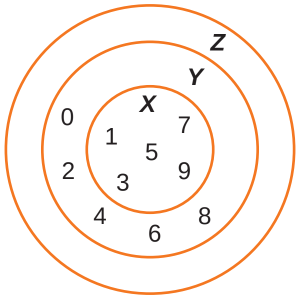
b. Since all the elements of Y are found in Z, and $n(Y) < n(Z)$, then $Y \subset Z$ is illustrated in the diagram on the right.



c. Since all the elements of X are contained in Z, and $n(X) < n(Z)$, then $X \subset Z$ is illustrated in the diagram on the right.



In example 2, note that $X \subset Y$ and $Y \subset Z$. This implies that $X \subset Z$, which is verified in example 2c. Thus, the relation among the three sets is illustrated as follows:



This shows the *transitive property of set inclusion*.

Key Concept

Transitive Property of Set Inclusion

For any sets A, B, and C, if $A \subset B$ and $B \subset C$, then $A \subset C$.

Consider another set W, which is the set of whole numbers. How do you compare Z and W? Which has more elements? In this case, $Z = W$ and $n(Z) = n(W)$. Hence, Z is said to be an *improper subset* of W.

Key Concept

For any sets A and B, A is said to be an **improper subset** of B if $A = B$.

Since any set is equal to itself, you can also conclude that $A \subseteq A$ and $B \subseteq B$. This property is referred to as the *reflexive property of set inclusion*.

Key Concept

Reflexive Property of Set Inclusion

For any set A, $A \subseteq A$.

The symbol $\subseteq$ is nonstrict, while the symbol $\subset$ is strict. Note that if $A \subseteq B$, then either $A \subset B$ or $A = B$.

Another subset of any set A is the empty set; that is, $\varnothing \subset A$. This means that $\varnothing$ is a *trivial subset* of any set. Is $\varnothing \subseteq \varnothing$?

Remember that any set has at least one of these types of subsets—improper, proper, or trivial subset. Now you will study the subsets that a given set could have.

EXAMPLE 3

Let $S = \{a, b, c\}$. How many subsets does S have? What are these subsets?

Answer:

Look for the improper, proper, and trivial subsets of set S.

Type of Subsets		Subset
Improper subset	The improper subset of any set is itself.	$\{a, b, c\}$
Proper subset	One-element subsets	$\{a\}, \{b\}, \{c\}$
	Two-element subsets	$\{a, b\}, \{b, c\}, \{a, c\}$
Trivial subset	The empty set is a trivial subset of any set.	$\varnothing$

Thus, S has 8 subsets, namely: $\{a, b, c\}$, $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, and $\{a, c\}$.

How many subsets does a set with 4 elements have? Try answering the question using the set $T = \{a, b, c, d\}$. Use an organized way of finding all the subsets so you will not miss out any subset.

Disjoint Sets

Look at the following sets:

$M = \{1, 3, 5, 7, 9\}$ $N = \{2, 4, 6, 8, 10\}$ $O = \{1, 2, 3, 4, 5\}$

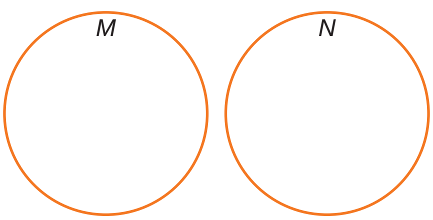
In what way can you show the relations among sets M, N, and O?

Notice that sets M and N have no common elements. Thus, M and N are *disjoint sets*.

Key Concept

Two sets are said to be **disjoint sets** if they have no common element.

The disjoint sets M and N are illustrated in the following diagram:



Based on the given sets, can you say that M and O are disjoint?

M and O have the following common elements: 1, 3, and 5. Therefore, M and O are not disjoint.

EXAMPLE 4

Consider the following sets:

$O = \{o \mid o \text{ is an odd number}\}$

$E = \{e \mid e \text{ is an even number}\}$

$C = \{0, 1, 2, 3, 4, 5\}$

Tell whether or not each pair of sets are disjoint. If the two sets are not disjoint, identify their common element(s).

a. O and C b. O and E c. E and C

Answers:

a. O and C are not disjoint sets. They both contain 1, 3, and 5.

b. O and E are disjoint sets.

c. E and C are not disjoint sets. Their common elements are 0, 2, and 4.

PRACTICE 1-2

Concepts and Skills

Write *True* if the statement is true; otherwise, write *False*.

1. An empty set is a subset of itself.
2. Any two equivalent sets are *not* disjoint sets.
3. An empty set has *no* proper subset.
4. If $A \subset B$, then A and B are *not* disjoint.
5. An element of a set can be the same as another set.
6. The improper subset of any set is equal to the empty set.
7. A subset of B is a subset of C only if B is a subset of C.
8. If A and B are both equivalent to C, then A and B are equal.
9. If B is a proper subset of A, then B does *not* necessarily contain each element of A.
10. If A and B are equal to C, then A and B are *not* necessarily equal.

Refer to the following sets to answer items **11** to **20**:

A = $\{x|\, x$ is an even digit$\}$
B = $\{x|\, x$ is a multiple of 2 from 0 to 8$\}$
C = $\{x|\, x$ is a prime number from 2 to 11$\}$
D = $\{x|\, x$ is an even prime number$\}$
E = $\{x|\, x$ is a counting number between 2 and 11$\}$
F = $\{x|\, x$ is a whole number less than 10 that is a multiple of 10$\}$

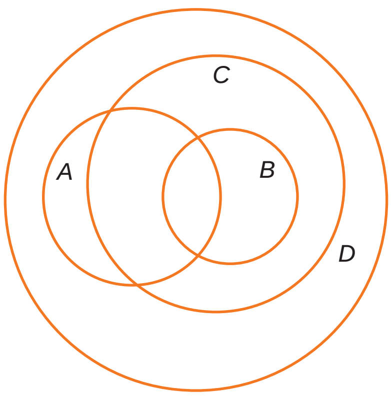
Write *True* if the statement is true; otherwise, write *False*.

11. $A \sim B$
12. $B \neq C$
13. $F = \varnothing$
14. $D \not\subset B$
15. $C \subset E$
16. C and A are disjoint sets.

From the six previously given sets, choose the sets that are being described in each item.

17. two sets that are both equal and equivalent
18. two sets that are equivalent but *not* equal
19. three sets that are subsets of B
20. a subset of A that is *not* a subset of E

For items **21** to **28**, refer to the following diagram:



Write True if the relation is true; otherwise, write False.

21. $A \subset B$
22. $A \subset D$
23. $B \subset C$
24. $D \subset C$
25. $A \subset C \subset D$
26. $B \subset A \subset C$
27. $B \subset C \subset D$
28. $D \supset C \supset B$

For items **29** to **37**, consider the set $V = \{0, 10, \{100\}\}$. Write True if the statement is true; otherwise, write False and modify the relation symbol to make it correct.

29. $0 \in V$

30. $\varnothing \in V$

31. $10 \subset V$

32. $\{10\} \subset V$

33. $\{100\} \in V$

34. $\{10\} \subset \{100\}$

35. $\{0\} \in V$

36. $\{\{100\}\} = \{100\}$

37. $\{0\} = \varnothing$

For items **38** to **41**, let S be the set of distinct letters in the word "smiles." Identify all the subsets of S that satisfy each description.

38. subsets that have *no* element

39. subsets that contain only one vowel

40. subsets that do *not* contain a vowel

41. subsets that contain exactly three elements

Applications

Answer each item completely.

42. Your school is offering the following PE subjects: basketball, volleyball, table tennis, badminton, and baseball. Is the set of PE subjects offered in your school a subset of the set of sports played in the Summer Olympic Games? Explain.

43. Liza can choose three from the five elective subjects offered during a term. The elective subjects offered are Spanish, Leadership, Music Composition, Psychology, and Actuarial Science. What are the possible sets of elective subjects she can enroll in?

44. A science teacher divides his class into 8 groups for a project. He wants his students to research on the planets in the solar system. Can he assign a different planet to each group? What set concept is illustrated in this situation? Explain.

45. Five friends plan a chess tournament. Each person will play against each participant exactly once. How many possible games can be played?

Consider sets A, B, and C. Draw a Venn diagram that illustrates each statement.

46. $A \subset B$, but $B \not\subset C$

47. $A \subseteq B$ and $B \subseteq A$

48. $A \subset B$, $A \subset C$, and $B \subset C$

49. $A \subset C$ and $B \subset C$, but $A \not\subset B$

50. A and B are disjoint and $B \subset C$

Enrichment Exercises

In example 3 of this lesson, you have seen that a set with 3 elements has 8 subsets. Is there a relationship between the cardinality of a set and the number of subsets of the set? Investigate by answering each question.

51. How many subsets does an empty set have? Give the subsets.

52. What are the subsets of {0}? How many subsets does this set have?

53. What are the subsets of {0, 1}? How many subsets does this set have?

54. Suppose that set A is a finite set with 6 elements. How many subsets does A have? Complete the table on the right. Then find a pattern on the number of subsets of a certain set based on its cardinality.

Cardinality	Number of Subsets
0	
1	
2	
3	8
4	
5	
6	

55. Based on the results you obtained in the previous item, make a generalization about the number of subsets of a finite set A, with $n(A) = k$.

Use your generalization in the previous item to find the total number of subsets of each given set.

56. $Y = \{x \mid x$ is a weekday that starts with a vowel$\}$

57. $M = \{x \mid x$ is a distinct letter in the word "mathematics"$\}$

58. $C = \{x \mid x$ is a distinct consonant in the word "consonant"$\}$

59. $D = \{x \mid x$ is a two-digit number that is divisible by 11$\}$

60. $J = \{x \mid x$ is a prime number from 1 to 10$\}$

LESSON 1-3

Set Operations

Learning Objectives

At the end of the lesson, you should be able to:

- identify operations on sets
- perform operations on sets
- illustrate set operations using Venn diagrams

Probe and Learn

In a school sports fest, four teams will participate. Each team will be composed of students in each year level. The grade 7 students will be the *Green Team*. The grade 8 students will be the *Yellow Team*. The grade 9 students will be the *Red Team*. The grade 10 students will be the *Blue Team*.

The four teams will compete with each other in all the events except for a special event. In this event, only two teams will compete. The grade 7 and grade 10 students will be grouped to form the *Blue-Green Team*. The grade 8 and grade 9 students will be grouped to form the *Red-Yellow Team*.

Like in numbers, there are also operations on sets. You can perform a specific operation on two or more sets to come up with another set. Study each set operation.

Union of Sets

In the given situation, each team in the sports fest represents a set. Combining two teams represents a *union of two sets*. What can you say about the elements of the new set formed?

Key Concept

For any sets A and B, the **union of A and B**, which is denoted as $A \cup B$ (read as "A union B"), is given by:

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

The word "or" means that x belongs to A, or to B, or to both A and B. Hence, to find the union of two sets, combine the elements of A and B by writing the elements in any order, without repetition.

Example 1

Consider the following sets of athletes:

B = {Curry, Durant, James, Irving}

S = {Messi, Ronaldo, Suarez, Neymar}

T = {Yap, Tenorio, Pingris, Romeo}

Find the elements of each union of sets.

a. $S \cup T$

b. $B \cup S$

c. $B \cup T$

d. $T \cup B$

Solutions:

a. $S \cup T$ = {Messi, Ronaldo, Suarez, Neymar, Yap, Tenorio, Pingris, Romeo}

b. $B \cup S$ = {Curry, Durant, James, Irving, Messi, Ronaldo, Suarez, Neymar}

c. $B \cup T$ = {Curry, Durant, James, Irving, Yap, Tenorio, Pingris, Romeo}

d. $T \cup B$ = {Yap, Tenorio, Pingris, Romeo, Curry, Durant, James, Irving}

In the previous example, note that $B \cup T = T \cup B$. This illustrates that the union of sets is *commutative*; that is, the order of the sets does not affect the union.

Key Concept

Commutative Property of the Union of Sets

For any sets A and B,

$$A \cup B = B \cup A.$$

Study the next example.

EXAMPLE 2

Suppose that the four teams mentioned at the start of the lesson are described as follows:

G = {x | x is a grade 7 student who is a member of *Green Team*}

Y = {x | x is a grade 8 student who is a member of *Yellow Team*}

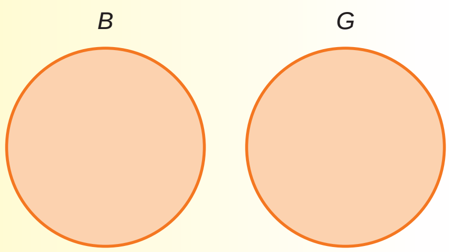
R = {x | x is a grade 9 student who is a member of *Red Team*}

B = {x | x is a grade 10 student who is a member of *Blue Team*}

Illustrate the *Blue-Green Team* using a Venn diagram.

Solution:

Note that set B refers to *Blue Team* and set G refers to *Green Team*. Thus, the *Blue-Green Team* refers to the union of B and G, or $B \cup G$. Observe that B and G are disjoint sets since they do not have any common element. This is because no grade 7 student is a grade 10 student, and no grade 10 student is a grade 7 student. Thus, the *Blue-Green Team* or $B \cup G$ is illustrated as follows:



$B \cup G$

Look at the Venn diagram in the previous example. The shaded region represents $B \cup G$, which is the combined region representing B and G. This means that the total number of students in the combined set is equal to the sum of the number of grade 7 students and the number of grade 10 students. This illustrates the following generalization:

Key Concept

If A and B are two disjoint finite sets, then $A \cup B$ is also finite and

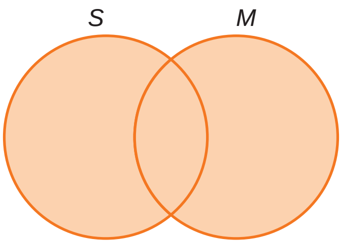
$$n(A \cup B) = n(A) + n(B).$$

Verify this fact using example 1.

Intersection of Sets

Suppose that in the sports fest, the presidents of the Science Club and the Math Club agree to unite their members. Since some students might be members of both clubs, the number of members in the union might be less than the sum of the original number of members of the two clubs.

Let S be the set of Science Club members. Let M be the set of Math Club members. Since there is a possibility of overlap between S and M, the sets may not be disjoint. In such case, the union of these sets may be illustrated as follows:



$S \cup M$

In this situation, the students who are members of both the Science Club and the Math Club form another set. This set is referred to as the *intersection* of S and M. The intersection of S and M is the set of all elements contained in both S and M.

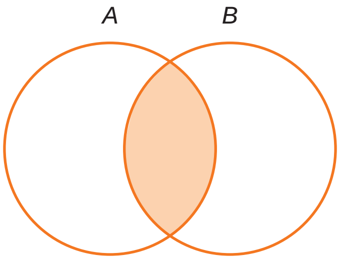
Key Concept

For any sets A and B, the **intersection of A and B**, which is denoted as $A \cap B$ (read as "A intersection B"), is given by:

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

The word "and" means that the elements belong to both A and B.

Using a Venn diagram, $A \cap B$ can be illustrated as follows:



$A \cap B$

In the diagram, the shaded region represents $A \cap B$, which is the set of all elements that are common to both A and B. Notice that the shaded region also represents $B \cap A$. This illustrates that the intersection of sets is also commutative.

Key Concept

Commutative Property of the Intersection of Sets

For any sets A and B,

$$A \cap B = B \cap A.$$

Complement of a Set

Another operation involving sets is the *set complement*. This operation is performed with respect to a *universal set*.

Key Concept

The **universal set** is the set that contains all the elements under consideration. This is usually denoted as U.

For instance, consider the set $V = \{x | x$ is a vowel in the English alphabet$\}$. Consider also set $L = \{x | x$ is a letter in the English alphabet$\}$. You know that set V is contained in set L. Thus, the universal set in this situation is $L = \{x | x$ is a letter in the English alphabet$\}$. Similarly, when considering the set $C = \{x | x$ is a consonant in the English alphabet$\}$, the universal set is L.

Study the following definition of the complement of a set:

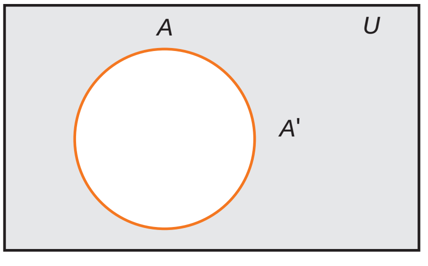
Key Concept

Let $A \subset U$. The **complement** of A is the set of elements of U that are not contained in A. It is denoted by A' (read as "A complement"). In symbols,

$$A' = \{x | x \in U \text{ and } x \notin A\}.$$

To find the complement of a set, look for the elements in the universal set that are not in the given set. For instance, to find A', remove all the elements of A from U. The elements that remain in U are the elements of A'.

The shaded region in the following diagram represents A'. In a Venn diagram, the universal set is represented by a rectangle. Inside the rectangle are the circles that represent the sets involving the universal set.



Consider again the sets V and L. Suppose that the vowels are removed from the set of the letters in the English alphabet. What letters remain? The set of the remaining letters represent V'. Thus, V' contains the consonants. Since $C = \{x | x \text{ is a consonant in the English alphabet}\}$, then $V' = C$. Conversely, $C' = V$.

EXAMPLE 3

Consider the following sets:

$U = \{x | x \text{ is a digit in the decimal system}\}$
$S = \{x | x \text{ is an even whole number less than } 10\}$
$T = \{x | x \text{ is a prime number less than } 10\}$

Describe each set using the roster method.

a. S' b. T' c. U'

Solutions:

First, describe sets U, S, and T using the roster method to perform the operation more easily.

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
$S = \{0, 2, 4, 6, 8\}$
$T = \{2, 3, 5, 7\}$

a. For S', take away the elements of set S from U.

$S' = \{\not{0}, 1, \not{2}, 3, \not{4}, 5, \not{6}, 7, \not{8}, 9\} = \{1, 3, 5, 7, 9\}$

b. $T' = \{0, 1, \not{2}, \not{3}, 4, \not{5}, 6, \not{7}, 8, 9\} = \{0, 1, 4, 6, 8, 9\}$

c. $U' = \{\not{0}, \not{1}, \not{2}, \not{3}, \not{4}, \not{5}, \not{6}, \not{7}, \not{8}, \not{9}\} = \varnothing$

Notice that for any set A combined with its complement A', the resulting union is the universal set. The following properties hold true for the complement of a set:

Key Concept

For any set A and a universal set U:

$$A \cup A' = U$$

$$A \cap A' = \varnothing$$

Verify these properties using Venn diagrams.

Set Difference

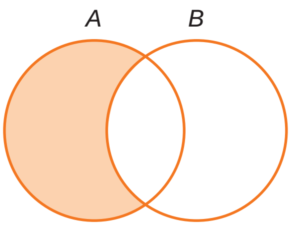
Another operation that is similar to set complement is *set difference*. Both operations involve "taking away" elements from a set. To find the complement of a set, take away the elements of that set from the universal set. To find the difference, you will take away the elements of one set that are in the other set. The set difference is defined as follows:

Key Concept

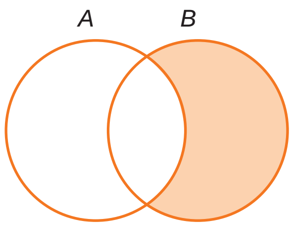
The **difference** of sets A and B, which is denoted as $A - B$, is the set of elements of A that are not in B. In symbols,

$$A - B = \{x|x \in A \text{ and } x \notin B\}.$$

Set difference may be illustrated using Venn diagrams as follows:



$A - B$



$B - A$

What do you observe in the two diagrams?

Consider again sets S and T in example 3.

$S = \{0, 2, 4, 6, 8\}$

$T = \{2, 3, 5, 7\}$

Note that the only element of S that is in T is 2. Thus, to find $S - T$, take this element (2) away from S. Hence, $S - T = \{0,$ ~~2~~$, 4, 6, 8\} = \{0, 4, 6, 8\}$.

What elements of S will be removed from T to find $T - S$?

EXAMPLE 4

Let $J = \{a, b, c, d, e\}$ and $V = \{a, e, i, o, u\}$.

Define each set.

a. $J - V$

b. $V - J$

Solutions:

a. The elements *a* and *e* of set *J* are contained in *V*. Thus, to find $J - V$, take these away from set *J*. Therefore,

$$J - V = \{\not{a}, b, c, d, \not{e}\} = \{b, c, d\}.$$

b. $V - J = \{i, o, u\}$. Can you explain why?

Based on the previous example, can you say that set difference is commutative?

Set Product

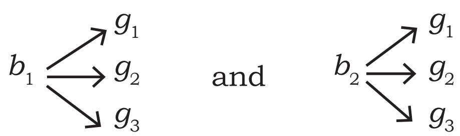
Consider the following situation:

Five players are planning their fencing matches, in such a way that each player must compete with another player of the opposite gender. There are 2 boys and 3 girls. How many matches will there be?

Do you know?

Fencing, also known as sword fighting, is a fantastic way to improve one's balance, coordination, and flexibility. In fencing, two competitors face off in a "bout." Fencers use blade work, footwork, and strategy to beat their opponents.

Let the set of boys be given by $B = \{b_1, b_2\}$. Let the set of girls be represented by $G = \{g_1, g_2, g_3\}$. To find the possible matches, pair each boy with each girl. This can be illustrated in a diagram as follows:



Based on the diagram, the possible pairs are (b_1, g_1), (b_1, g_2), (b_1, g_3), (b_2, g_1), (b_2, g_2), and (b_2, g_3).

Such pairing of the elements between two sets is called *set product*.

Key Concept

The **set product** or **Cartesian product** of two sets *A* and *B* is the set of all possible ordered pairs (*a*, *b*), where $a \in A$ and $b \in B$. In symbols, the set product $A \times B$ (read as "*A* cross *B*") is given by:

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

In the given situation,

$$B \times G = \{(b_1, g_1), (b_1, g_2), (b_1, g_3), (b_2, g_1), (b_2, g_2), (b_2, g_3)\}.$$

Therefore, the five players will play a total of 6 matches.

Note that the order of the elements in an ordered pair is important. The ordered pairs (a, b) and (c, d) are the same if and only if $a = c$ and $b = d$. For example, the ordered pair (1, 2) is different from (2, 1). Does this mean that $A \times B$ is not the same as $B \times A$? Study the next example.

EXAMPLE 5

Let $C = \{a, b, c\}$ and $D = \{1, 2, 3\}$. Define each set.

a. $C \times D$

b. $D \times C$

Solutions:

a. $C \times D = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$

b. $D \times C = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$

Based on the previous example, can you say that set product is commutative? Explain.

PRACTICE 1-3

Concepts and Skills

Determine whether each statement is *always*, *sometimes*, or *never* true.

1. $\varnothing' \subseteq U$

2. $A - B = B$

3. $A \cup B = A$

4. $A' \cap A = \varnothing$

5. $A \subseteq (A \cup B)$

6. $A \subseteq (A \cap B)$

7. $(A \cap B) \subseteq A$

8. $n(A) + n(A') = n(U)$

9. $n(A - B) = n(A) - n(B)$

10. $n(A) + n(B) = n(A \cup B)$

11. If $A \subseteq U$, then $A' \subseteq U$.

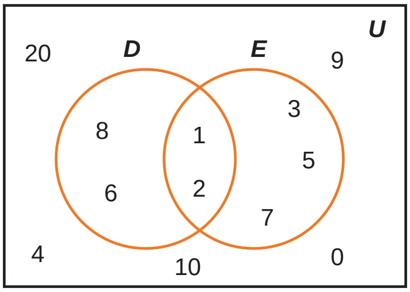
12. If $A \neq \varnothing$, then $\varnothing \cup A = \varnothing$.

13. If $A \subseteq U$, then $A' = U - A$.

14. If $A = B$, then $A \cup B = A \cap B$.

15. If $A \sim B$, then A and B are disjoint.

For items **16** to **25**, refer to the following diagram:



Perform the indicated operations.

16. $D \cap E$

17. D'

18. $D - E$

19. $(D \cup E)'$

20. $(E - D)'$

21. $D' \cup E'$

22. $D' \cap E'$

23. $D' \cup \varnothing'$

24. $E' \cap U$

25. $(D \cup E)' \times (D \cap E)$

For items **26** to **35**, consider sets A and U, where $A \subset U$. Perform the indicated operations.

26. $A \cup A$

27. $A \cap A$

28. $A \cup U$

29. $A \cap U$

30. $\varnothing \cap A$

31. $\varnothing \cap U$

32. $\varnothing' \cup U$

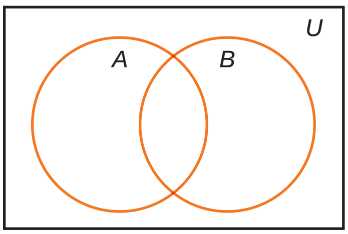
33. $\varnothing' \cup \varnothing$

34. $\varnothing' \cap U'$

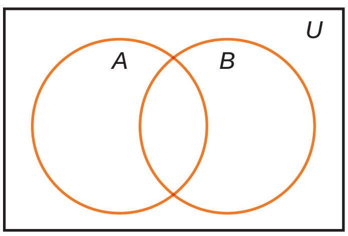
35. $U \cap A'$

Shade the region that represents each given set in each Venn diagram.

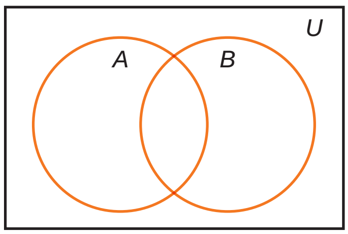
36. A



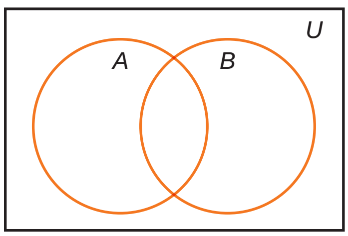
37. B'



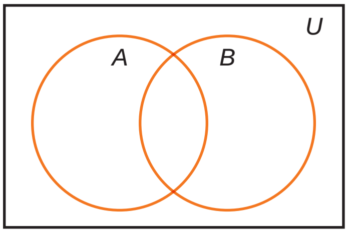
38. $U - A$



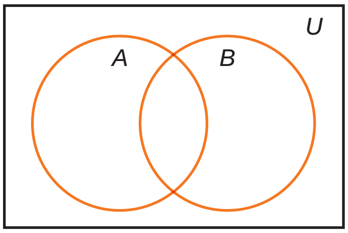
39. $A \cap B$



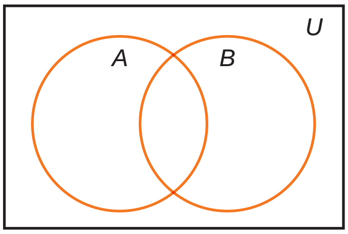
40. $(A \cap B)'$



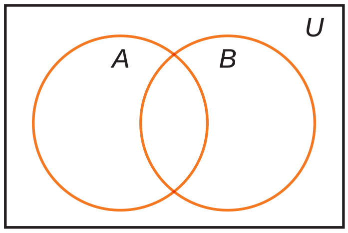
41. $A \cup B$



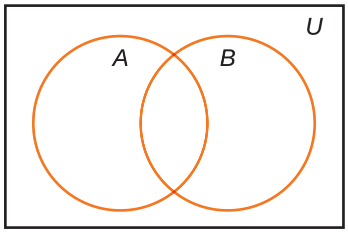
42. $(A \cup B)'$



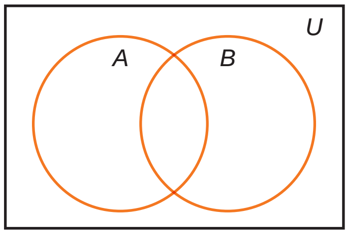
43. $A \cap B'$



44. $(B - A)'$



45. $U - A'$



Applications

Answer each item completely.

46. A canteen offers three kinds of sandwiches and two flavors of fruit shake. How many possible combinations of a sandwich and a fruit shake are there?

For items **47** to **50**, consider the following situation:

In a circle of 12 friends, 10 are varsity players. Ariel, Bart, Carlo, David, Eric, Frank, and George are members of the basketball team. Andrei, Brian, Carlo, Daniel, and Eric are members of the volleyball team.

47. Who are the members of either the volleyball team *or* the basketball team?

48. Who are the members of both the volleyball team *and* the basketball team?

49. Who are the members of the basketball team but are *not* members of the volleyball team?

50. Who are the members of the volleyball team but are *not* members of the basketball team?

Enrichment Exercises

Show your answer to each item using a Venn diagram.

51. If $A \subset B$, find $A \cap B$.

52. If $A \subset B$, find $A \cup B$.

53. If $A \subset B$ and $B \subset C$, find $A \cap C$.

54. If A and B are disjoint sets and $A \subset C$, find $A \cap C$.

55. If A and B are *not* disjoint sets and $C \subset (A \cap B)$, find $C \cap (A \cap B)$.

Answer each item.

56. Is $(A \cap B) \cap C = A \cap (B \cap C)$? Give an example to support your answer.

57. If A and B are disjoint sets, is $A - B = A$? Explain.

58. If A and B are disjoint sets, is $A \cap B = \varnothing$? Explain.

59. If A and B are finite sets that are *not* disjoint, is $n(A \cup B) = n(A) + n(B)$? Explain.

60. How does set difference differ from set complement?

LESSON 1-4

Problem Solving Involving Sets

Learning Objectives

At the end of the lesson, you should be able to:

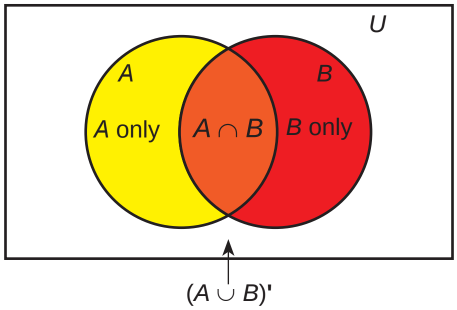
- illustrate problems involving sets using Venn diagrams
- solve problems involving sets

Probe and Learn

A teacher conducts a survey among 50 students. She asked them which of the two sports, basketball or volleyball, they would like to play. Each student can choose one or two sports, or none of the two sports if they do not like to play either sport. The teacher finds out that 8 students like to play both basketball and volleyball, 24 prefer volleyball, and 32 like basketball. How many students do not like basketball or volleyball?

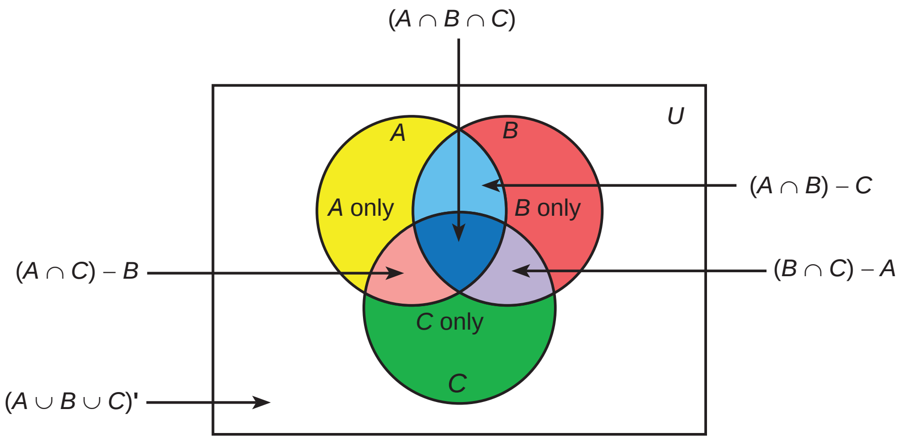
Many problems involving sets can be solved using Venn diagrams. Recall that a Venn diagram can help you visualize the logical relationships among sets. It is made by drawing a rectangle to represent the universal set. Inside the rectangle are two or more circles, where each circle represents a specific set. The circles divide the universal set into several regions, wherein each region represents a set of elements that share common properties. It is important that you know how to illustrate problems using Venn diagrams. Study the following illustrations:

A Venn diagram involving two sets *A* and *B* is shown.



The region where the two circles overlap represents $A \cap B$, as labeled in the diagram. It contains the elements that belong to both *A* and *B*. The region labeled "*A* only" represents those elements that belong to *A* but not to *B*. Similarly, the region labeled "*B* only" represents those elements that belong to *B* but not to *A*. On the other hand, the entire region covered by both *A* and *B*, which is the colored region, represents $A \cup B$. The uncolored region inside the rectangle represents $(A \cup B)'$.

Now study the logical relations among three sets *A*, *B*, and *C*, as shown here.



As you can see in the diagram, each region is specified. The entire colored region shows $A \cup B \cup C$.

A Venn diagram could clearly represent the logical conditions specified in a certain problem. That is why it is a useful tool for solving problems involving sets. You can now solve the problem posed at the start of the lesson. Study the solution in the next example.

EXAMPLE 1

Out of the 50 students surveyed by the teacher, how many students do *not* like basketball or volleyball?

Solution:

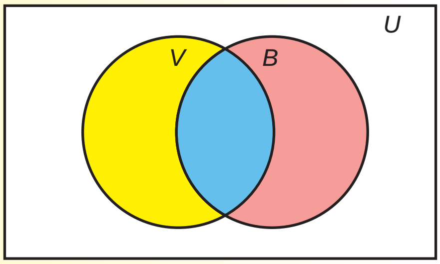
If U is the universal set, you can represent the sets in the problem as follows:

$U = \{x \mid x \text{ is a student who was surveyed}\}$

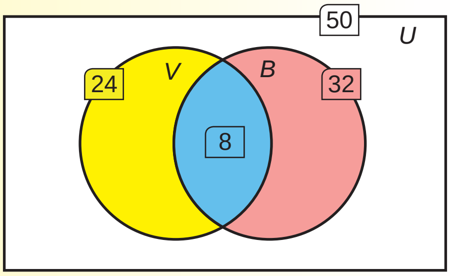
$V = \{v \mid v \text{ is a student who likes volleyball}\}$

$B = \{b \mid b \text{ is a student who likes basketball}\}$

To construct the Venn diagram for this problem, draw two circles (representing sets V and B) inside a rectangle (representing set U). Note that some students are elements of both V and B. This implies that V and B intersect. Thus, V and B are not disjoint. So you will have the following diagram:



In the following diagram, $n(U) = 50$, $n(V) = 24$, $n(B) = 32$, and $n(V \cap B) = 8$



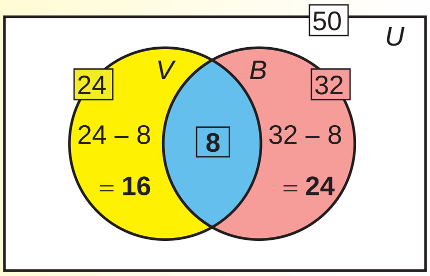
The next step is to determine the number of students who like volleyball only and those who like basketball only. Note that the number of students who like volleyball only refers to $n(V - B)$. The number of students who like basketball only refers to $n(B - V)$.

Out of 24 elements in V, 8 elements belong to B. Thus,

$$n(V - B) = 24 - 8 = 16.$$

This means that 16 students like volleyball only. Similarly, subtract 8 from 32 to find $n(B - V)$. So the number of students who prefer basketball only is:

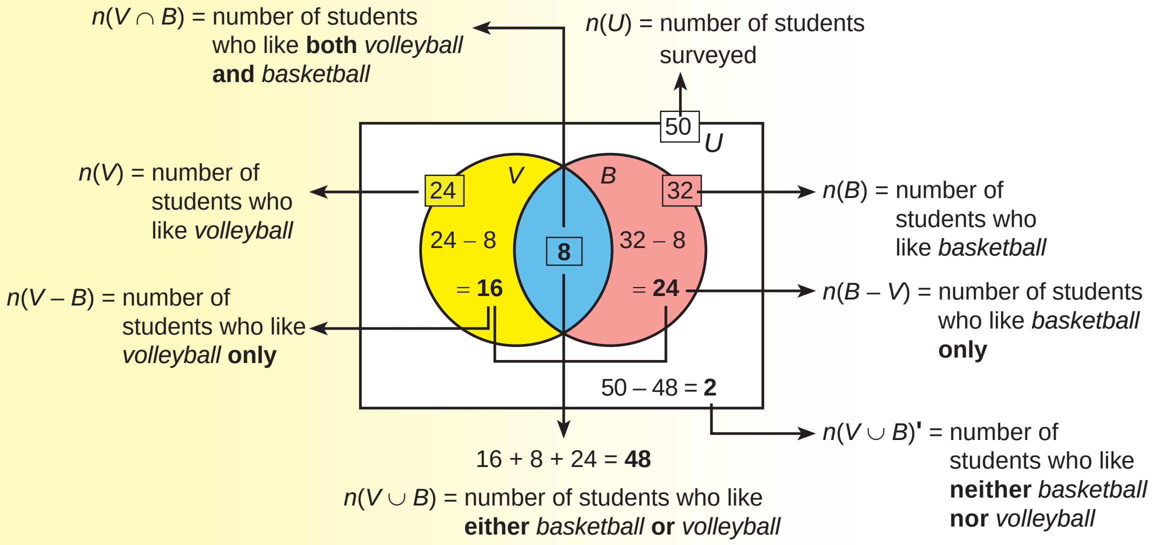
$$n(B - V) = 32 - 8 = 24.$$



Adding the number of students in the shaded regions gives the number of students who like either volleyball or basketball. Hence, there are 16 + 8 + 24 = 48 students who like either volleyball or basketball. This number corresponds to $n(V \cup B)$. So $n(V \cup B)'$ represents the number of students who do not like volleyball or basketball. Since the universal set has a total of 50 students, then:

$$n(V \cup B)' = 50 - 48 = 2.$$

Therefore, 2 of the students surveyed do not like any of the two sports. The complete Venn diagram is shown.



As shown in the previous example, a Venn diagram organizes data and shows the relationships between sets. Other information can also be extracted from the diagram, like the number of students who like only one sport.

If you will add the 24 students who prefer volleyball and the 32 students who like basketball, you will get 56 students. Note that this sum is greater than the number of students surveyed, which does not satisfy the given information in the problem. What is wrong with this solution?

Consider the 8 students who prefer both sports. These 8 students are included in both the number of students who like basketball and the number of students who like volleyball; thus, they are counted twice. Hence, you have to subtract 8 from the sum (56) to get a difference of 48. Consequently, there are 50 – 2 = 48 students who like either sport. Note that you obtained the same answer (2) as in the previous solution. In symbols, you may solve for the number of students who like either sport as follows:

$$\begin{array}{ccccccccc} 24 & + & 32 & - & 8 & = & 48 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ n(V) & + & n(B) & - & n(V \cap B) & = & n(V \cup B) \end{array}$$

This equation illustrates the counting formula for sets, which can be stated as follows:

Key Concept

If A and B are finite sets, then:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

An equivalent equation to this is:

$$n(A \cap B) = n(A) + n(B) - n(A \cup B).$$

EXAMPLE 2

A teacher asks the same set of students in example 1 which racket sport (badminton, lawn tennis, table tennis) they would like to play. Each student chooses at least one of the three racket sports, or none if they do not like to play any of the given sports. The results are as follows:

- 6 students like the three racket sports.
- 30 students like badminton.
- 22 students like lawn tennis.
- 25 students like table tennis.
- 10 students like both table tennis and lawn tennis.
- 15 students like both table tennis and badminton.
- 11 students like both lawn tennis and badminton.

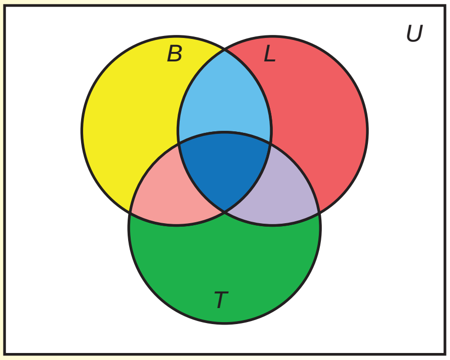
How many students do *not* like any of the three racket sports?

Solution:

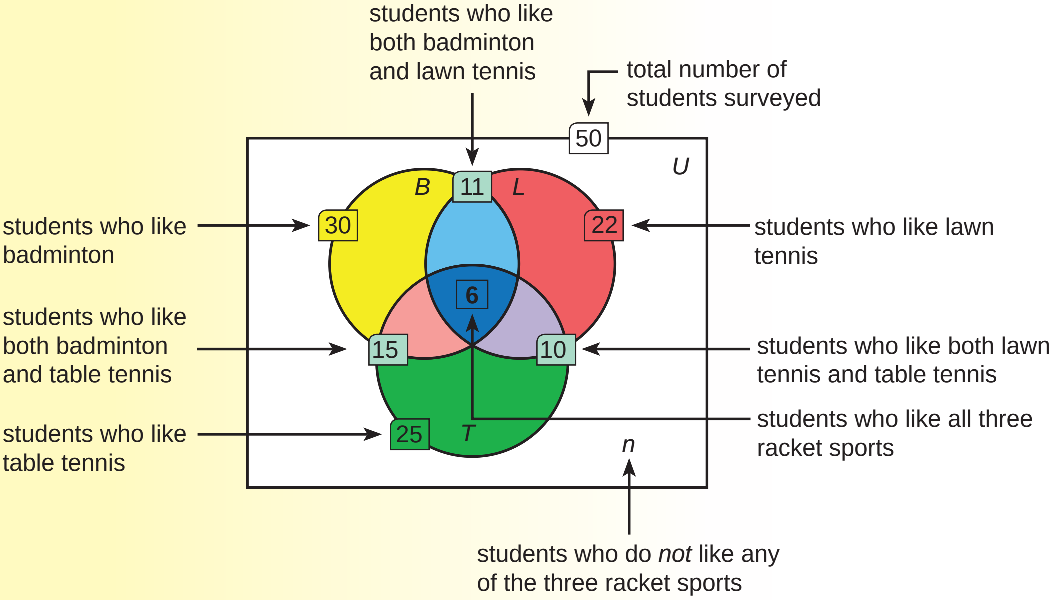
Three different sets of students can be formed. So you will have the following sets:

$B = \{b \mid b$ is a student who likes badminton$\}$
$L = \{w \mid w$ is a student who likes lawn tennis$\}$
$T = \{t \mid t$ is a student who likes table tennis$\}$

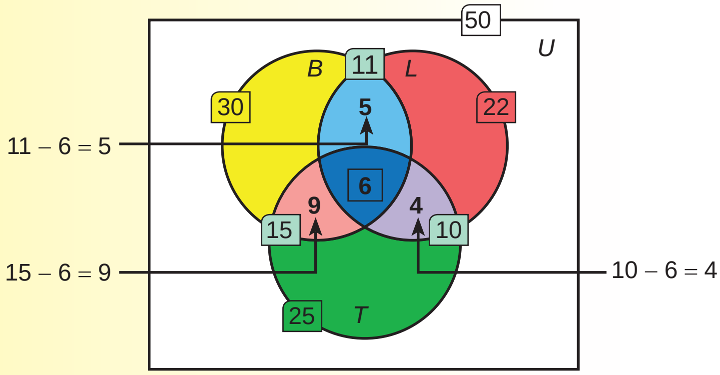
The universal set U is the same set of students surveyed in example 1. The following Venn diagram illustrates the relationships among sets B, L, and T. You can see that the three sets are represented by overlapping circles. Can you explain why?



Based on the given information, you can describe each region as follows:



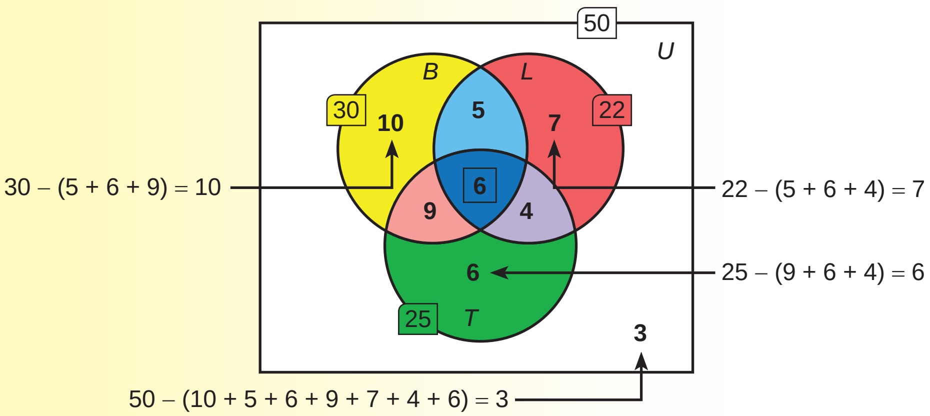
The diagram indicates the number of elements in each labeled set. Consider first the cardinality of the intersection of any two sets. In each of the boxed numbers in the diagram, the 6 students who like the three racket sports are already counted. Thus, subtract 6 from the boxed number in each of the three intersections.



Next solve for the numbers in the unfilled regions in each circle. It is given that the total number of elements in *B* is 30. So far, there are already 5 + 6 + 9 = 20 elements in *B* that have been accounted for. This implies that there are 10 elements in the unified region of *B* only. Do the same process for sets *L* and *T*.

Lastly consider the universal set that contains 50 students. To get the total number of students already indicated in the colored regions, you will have 10 + 5 + 6 + 9 + 7 + 4 + 6 = 47 students. Therefore, there are 3 students (that is, 50 – 47 = 3) who do not belong to sets *B*, *L*, and *T*.

The following is the complete Venn diagram for the problem:



Hence, 3 students do not like to play any of the three racket sports.

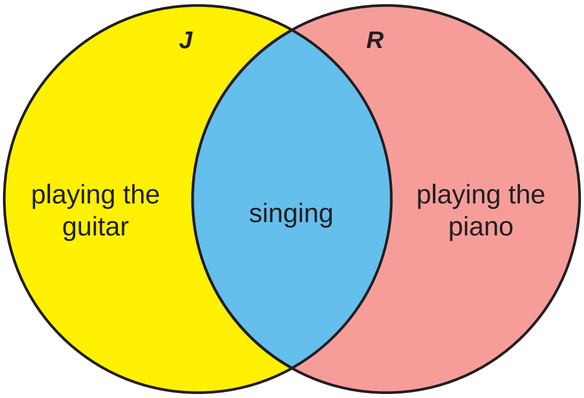
What other conclusions can you make from the diagram in example 2?

PRACTICE 1-4

Concepts and Skills

For items **1** to **8**, refer to the following situation:

Jay and Ray are both music enthusiasts. The following Venn diagram shows their interests, where J is the set of interests of Jay and R is the set of interests of Ray:



Based on the diagram, write True if the statement is true; otherwise, write False.

1. Jay likes singing.

2. Ray does *not* like singing.

3. Both Ray and Jay like singing.

4. Jay likes singing while Ray likes playing the piano.

5. Ray and Jay have exactly the same interests.

6. Ray and Jay like singing, playing the guitar, or playing the piano.

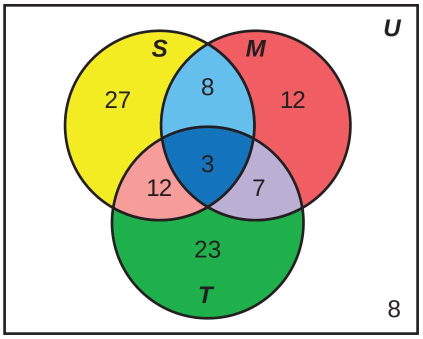
7. Both Jay and Ray like singing, while Ray likes playing the guitar.

8. Ray likes both singing and playing the guitar but he does *not* like playing the piano.

For items **9** to **25**, refer to the following text and diagrams:

A guidance counselor conducts a survey among her students about their membership in three clubs—Science Club, Math Club, and Theater Club. The results of the survey are shown in the Venn diagram on the right, where:

$S = \{s \mid s$ is a Science Club member$\}$;
$M = \{m \mid m$ is a Math Club member$\}$; and
$T = \{t \mid t$ is a Theater Club member$\}$.



9. How many students are members of all the three clubs?
10. How many students are *not* members of any of the three clubs?
11. How many students are members of the Science Club?
12. How many students are members of the Math Club?
13. How many students are members of the Theater Club?
14. How many students are members of both the Math Club and the Science Club?
15. How many students are members of both the Math Club and the Theater Club?
16. How many students are members of both the Science Club and the Theater Club?
17. How many students are members of the Math Club but *not* of the Theater Club and the Science Club?
18. How many students are members of the Science Club but *not* of the Theater Club and the Math Club?
19. How many students are members of the Theater Club but *not* of the Math and the Science Club?
20. How many students are members of both the Math Club and the Theater Club but *not* of the Science Club?
21. How many students are members of both the Science Club and the Theater Club but *not* of the Math Club?
22. How many students are members of both the Science Club and the Math Club but *not* of the Theater Club?
23. How many students did the guidance counselor survey?
24. How many students are members of exactly one club?
25. How many students are members of exactly two clubs?

Applications

Answer each item completely.

For items **26** to **30**, refer to the following situation:

In an aquarium, 20 different freshwater fishes are either hardy or resistant to diseases. Twelve are hardy and 10 are resistant to diseases.

26. Draw a Venn diagram to illustrate the given information.

27. How many fishes are both hardy and resistant to diseases?

28. How many fishes are *not* hardy?

29. How many fishes are *not* resistant to diseases?

30. How many fishes are hardy but are *not* resistant to diseases?

For items **31** to **35**, refer to the following situation:

In a class of 120 students, 45 students are members of dance clubs and 78 students are members of sports teams. On the other hand, 32 students are not members of any of the mentioned groups.

31. Draw a Venn diagram to illustrate the given information.

32. How many students are members of both dance clubs and sports teams?

33. How many students are *not* members of sports teams?

34. How many students are members of dance clubs but are *not* members of sports teams?

35. How many students are either members of dance clubs or members of sports teams?

For items **36** to **40**, refer to the following situation:

The science club adviser asks some student members what science courses they like. Twenty-eight members say they like physics, 35 say they like chemistry, and a 34 say they like biology. Also, 10 members say they like physics and chemistry, 12 say they like biology and chemistry, 9 say they like physics and biology, and 4 say they like all three. On the other hand, 10 say they do *not* like any of the three science courses.

36. Draw a Venn diagram to illustrate the given information.

37. How many students did the science club adviser ask?

38. How many students like both physics and chemistry but *not* biology?

39. How many students like any of the three courses?

40. How many students do *not* like physics?

For items **41** to **45**, refer to the following situation:

A class adviser is arranging the schedules of foreign language classes for 70 students. Thirty students prefer to study French, 30 prefer Spanish, and 30 prefer Nihongo. Twelve students prefer to study both French and Nihongo, and of these students, 5 prefer to study Spanish as well. Sixteen students prefer to study Nihongo only, and 20 prefer Spanish only.

41. Draw a Venn diagram to illustrate the given information.

42. How many students do *not* prefer to study Spanish, French, or Nihongo?

43. How many students prefer to study Spanish and Nihongo but *not* French?

44. How many students prefer to study French and Spanish but *not* Nihongo?

45. How many students prefer to study French only?

Enrichment Exercises

Answer each item completely.

For items **46** and **47**, refer to the following situation:

Suppose that 25 basketball players, 30 volleyball players, and 28 soccer players compose a varsity club.

46. If no member plays at least two sports, how many members are there in the varsity club?

47. Suppose that 5 basketball players are also volleyball players. No soccer player is also a volleyball or a basketball player. How many members are there in the club?

For items **48** and **49**, refer to the following situation:

Jose asked his classmates which telecommunication company they subscribed to. He found out that 14 of his classmates subscribed to company A and 18 subscribed to company B. He also found out that a total of 25 students subscribed to either company A or company B.

48. Can you tell how many students Jose asked? Explain.

49. What other information can you infer from the given data?

50. Let A, B, and C be disjoint finite sets. Is $n(A \cup B \cup C) = n(A) + n(B) + n(C)$? Give an example to support your answer.

Chapter Assessment

Write *True* if the statement is true; otherwise, write *False*.

1. If $S = \{0, 1\}$, then $\{0\} \in S$.

2. An empty set has no cardinality.

3. Any subset of a finite set is finite.

4. Any set is a proper subset of itself.

5. The set of points on a line is an infinite set.

6. The union of two sets is a set that contains their common elements.

7. The complement of a set is a subset of the universal set.

8. Two sets are equal if they have the same number of elements.

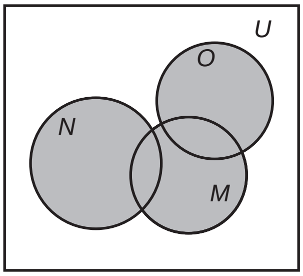
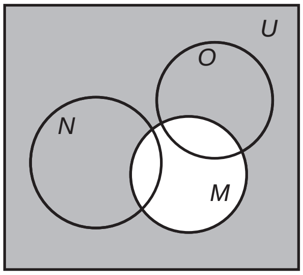
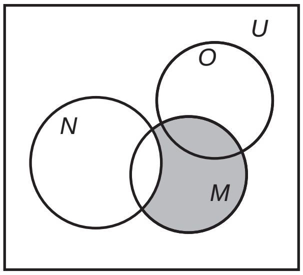
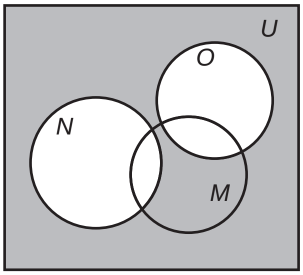
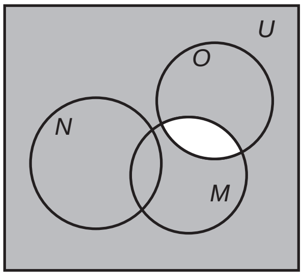
9. Two sets A and B are disjoint if no element of A is an element of B.

10. The elements of a set product are the products of the elements of the sets.

Choose the set relation in column B that corresponds to each pair of sets in column A.

Column A	Column B
11. $A = \{g, o, d\}$ and $B = \{l, o, v, e\}$	**A.** $A \sim B$
12. $A = \{g, o, d\}$ and $B = \{f, a, i, t, h\}$	**B.** $A = B$
13. $A = \{m, a, t, h\}$ and $B = \{l, o, v, e\}$	**C.** $A \subset B$
14. $A = \{m, a, t, h\}$ and $B = \{b \mid b$ is a distinct letter in "mahatma"$\}$	**D.** $A \cap B = \{o\}$
15. $A = \{m, a, t, h\}$ and $B = \{x \mid x$ is a distinct letter in "mathematics"$\}$	**E.** $A \cap B = \varnothing$

Choose the diagram in column B that illustrates each set in column A.

Column A	Column B
16. M'	**A.** 
17. $M - N$	**B.** 
18. $(N \cup O)'$	**C.** 
19. $(O \cap M)'$	**D.** 
20. $M \cup N \cup O$	**E.** 

Answer each question.

21. Which is *not* a set?
A. {m, a, t, h, e, i, c, s}
B. $\{x|x$ is a very large counting number$\}$
C. $\{x|x$ is a month with 32 days$\}$
D. $\{x|x$ is a bacterium in the human body$\}$

22. What method could be more appropriate to describe the set whose elements are Monday, April, 1, fish, apple, and love?
A. rule method
B. roster method
C. either A or B
D. neither A nor B

23. Which set is infinite?
A. all the points on a line
B. all the people on Earth
C. all the stars in the Milky Way Galaxy
D. all the trees in the Black Forest

24. Let G = {g, o, d} and $H = \{x|x$ is a distinct letter in the word "good"$\}$. Which relation is *not* true?
A. $G \subseteq H$
B. $G \sim H$
C. $G = H$
D. $G' = H$

25. Let $W = \{0, 1, 2, 3\}$. Which statement is true?
A. $\{\} \in W$
B. $0 \subset W$
C. $\{0\} \subset W$
D. $\{0\} \in W$

26. How many subsets does an empty set have?
A. 0
B. 1
C. 2
D. 3

27. Let $D = \{a, b, c, d\}$. How many subsets of D contain exactly two elements?
A. 2
B. 4
C. 6
D. 8

28. Which set operation is performed when combining all the elements of two or more sets?
A. intersection of sets
B. set difference
C. set product
D. union of sets

29. Let $U = \{x|x$ is a rainbow color$\}$ and S = {blue, green, yellow, red}. Which set is the complement of S?
A. {orange, pink, red}
B. {orange, pink, violet}
C. {orange, indigo, violet}
D. {red, blue, yellow, green}

30. Suppose that S = {s, m, i, l, e} and J = {j, o, v, i, a, l}. What is the cardinality of $S \times J$?
A. 8
B. 10
C. 25
D. 30

For items **31** to **35**, refer to the following situation:

The first term of a sequence is 2. Its second term is 4. The succeeding terms can be obtained by getting the average of all the preceding terms. Recall that the average is equal to the sum of the terms of a set divided by the number of terms. For example, the third term is the average of 2 (the first term) and 4 (the second term). It is given by:

$$\frac{\text{sum of the terms}}{\text{number of terms}} = \frac{2+4}{2} = \frac{6}{2} = 3$$

31. Let S be the set of all distinct terms in the sequence. Describe the set using the roster method.

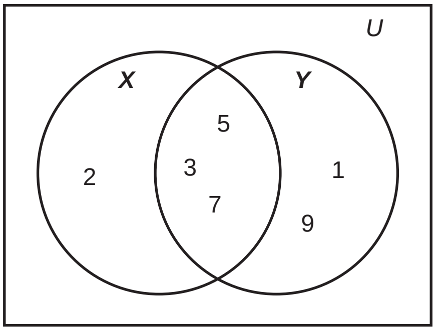
32. Find the cardinality of S.

33. Classify S as a finite set or an infinite set.

34. Find the number of subsets of S.

35. Let T be the set of the first ten counting numbers. Is $S \subset T$? Explain.

For items **36** to **40**, refer to the following diagram:



Let $U = \{x \mid x \text{ is a counting number less than } 10\}$.

Perform the indicated operations.

36. $X \cap Y$

37. $X - Y$

38. X'

39. Y'

40. $(X \cup Y)'$

For items **41** to **50**, refer to the following situation:

A teacher organizes a musical play. A total of 50 students will participate in the play. Suppose that 24 of the students will sing, 25 will act, and 23 will dance. Also, 16 of the students will sing and dance, 12 will act and dance, 14 will sing and act, and 6 will sing, act, and dance. Those students who will not perform will create the props for the play.

41. Draw a Venn diagram to illustrate the given information.

42. How many students will sing, act, or dance?

43. How many students will only act?

44. How many students will only sing?

45. How many students will only dance?

46. How many students will act and dance but will *not* sing?

47. How many students will sing and act but will *not* dance?

48. How many students will sing and dance but will *not* act?

49. How many students will sing or act but will *not* dance?

50. How many students will make the props for the play?

Answer each item completely.

51. Let $C = \{x \mid x$ is a counting number divisible by 2 or by 3$\}$. Let U be the set of counting numbers. Does C' contain a multiple of 9? Explain.

52. Let $A = \{1, 3, 5, 7, 9, 11\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. Which elements of A should be removed so that $A \cap B = \varnothing$?

53. In a school of 1,400 students, $\frac{2}{7}$ are grade 7 students. Two-fifths of the grade 7 students are girls. Also, 75 of the girls are into swimming and 87 are into gymnastics. Suppose that all the girls are enrolled in at least one of the two sports. How many girls are into both swimming and gymnastics?

54. A nurse surveyed 26 children. She learned that 14 of the children have dogs, 9 have cats, and 6 have rabbits. Also, 4 have dogs and cats, 3 have dogs and rabbits, and 1 has cats and rabbits. Suppose that none of them have all three kinds of pets. How many children do *not* have any of the three kinds of pets?

55. In a certain municipality, a market analyst asked 150 mothers about the detergent brand they use. The analyst found out that 102 of the mothers use brand A, 70 use brand B, and 40 use brand C. He also found out that 25 use brand A and brand B, 27 use brand A and brand C, and 30 use brand C and brand B. Suppose that all of them use at least one of the three brands. Are there mothers who use all three brands? Show your answer using a Venn diagram.

CHAPTER 2

The Real Number System

Look closely at the head of the sunflower. You would notice two spirals winding in two directions. The number of spirals in each direction is not the same—some sunflowers have 21 and 34. Others have 34 and 55, 55 and 89, or 89 and 144. These numbers of spirals belong to a special sequence of numbers, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ..., called the *Fibonacci sequence*. In this sequence, the next number is found by adding the two numbers preceding it. Fibonacci numbers can also be observed in plants, petals, tree branches, and fruits. On a pineapple, for instance, there are 8 scales in a diagonal in one direction and 13 in another. If you count the number of petals in a flower, you will most likely get a Fibonacci number.

Mathematics is the language of nature. In this chapter, you will learn about the different types of numbers and their properties, as you discover the beauty of math and nature.

LESSON 2-1

Real Numbers

Learning Objectives

At the end of the lesson, you should be able to:

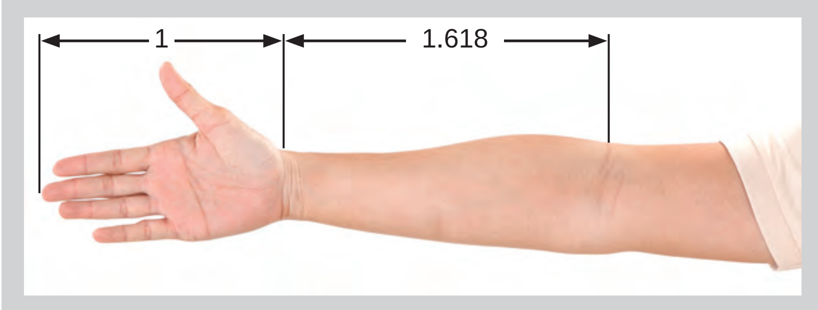
- apply the properties of real numbers
- identify the subsets of real numbers
- classify real numbers

Probe and Learn

A unique property of Fibonacci numbers is seen when dividing any two successive numbers. Look at the quotient of each pair of successive numbers (A and B) in the third column $\left(\frac{B}{A}\right)$ of the following table:

A	B	$\frac{B}{A}$
2	3	1.5
3	5	1.666666666...
5	8	1.6
8	13	1.625
⋮	⋮	⋮
144	233	1.618055556...
233	377	1.618025751...
⋮	⋮	⋮

Note that $\frac{B}{A}$ gets closer and closer to a special number called the *golden ratio*, which has a value of 1.6180339887.... The golden ratio is denoted by the Greek letter phi φ. Do you know that the golden ratio can also be observed on the characteristics of many things around you? Different parts of the human body also show the golden ratio. For instance, the ratio of the length of the forearm to the length of the hand is equal to the value of ϕ.



The golden ratio is an example of an irrational number. The set of irrational numbers is a subset of the set of *real numbers*.

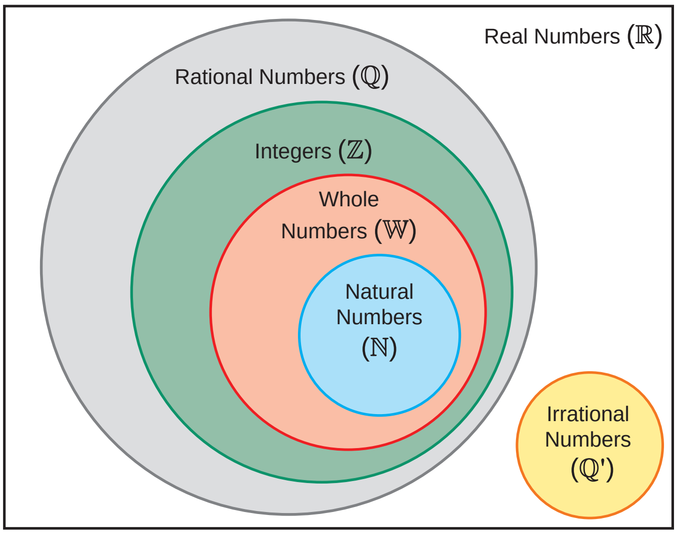
Set of Real Numbers

The set of *real numbers* (ℝ) has many subsets. Refer to the following list of subsets of ℝ:

- Numbers that are used in counting objects comprise the set of *natural numbers* (ℕ). So, ℕ = {1, 2, 3, ...}.
- Natural numbers and 0 (zero) comprise the set of *whole numbers* (𝕎). So, 𝕎 = {0, 1, 2, 3, ...}.
- Natural numbers, their negatives, and 0 comprise the set of *integers* (ℤ). So, ℤ = {..., –3, –2, –1, 0, 1, 2, 3, ...}.
- Numbers that can be expressed as a ratio of two integers belong to the set of *rational numbers* (ℚ). Rational numbers can be terminating decimals (like 0.5 and 0.75), repeating decimals (like 0.333... and 0.2727...), or fractions $\left(\text{like } \frac{1}{2} \text{ and } \frac{3}{4}\right)$.
- Nonterminating and nonrepeating decimals (like 1.6180339887... and 3.1416...) belong to the set of *irrational numbers* (ℚ').

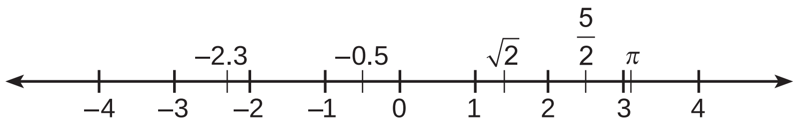
The Venn diagram on the right can help you visualize the relationships among the set of real numbers and its subsets.

Using your knowledge of sets, what generalizations can you make from the given diagram?



The Real Number Line

An important characteristic of the set of real numbers is that they can be represented by the points on a number line. In fact, the graph of the set of real numbers is the entire number line. Every point on the number line corresponds to a unique real number. Look at the following number line:



The numbers on the right of 0 are positive real numbers. The numbers on the left of 0 are negative real numbers. Some real numbers shown on the number line are the positive integers 1, 2, 3, and 4, the negative integers –1, –2, –3, and –4, the rational numbers –2.3, –0.5, and $\frac{5}{2}$, and the irrational numbers $\sqrt{2}$ and π.

Properties of Real Numbers

The *real number system* consists of the set of real numbers and the numbers that result from binary operations such as the four fundamental operations. Several assumptions govern the operations on real numbers. These assumptions are called *axioms*.

Key Concept

Axioms are properties that are assumed to be true without proof.

Study the following axioms for real numbers:

Key Concept

Closure Property

Let *a* and *b* be real numbers.

Axiom 1. *Closure Property of Addition*

The sum $a + b$ is a real number.

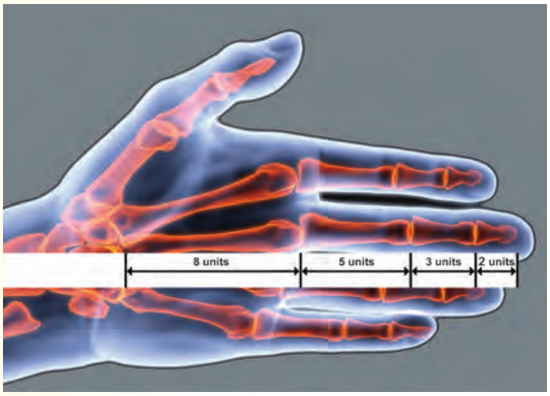
Axiom 2. *Closure Property of Multiplication*

The product $a \cdot b$ is a real number.

A set of numbers is said to be *closed under an operation* if, upon performing the operation on any two elements of the set, the result is also an element of that set. The closure property for real numbers guarantees that the sum and the product of any two real numbers are also real numbers. Thus, the set of real numbers is said to be closed under addition and multiplication.

EXAMPLE 1

Each finger of a human hand is divided into four parts by the knuckles. The relative lengths of these four parts are shown on the right. It could be a coincidence, but these four numbers, 2, 3, 5, and 8 belong to the Fibonacci sequence. Suppose that $F = \{2, 3, 5, 8\}$. Is F closed under addition?



Solution:

To check if set F is closed under addition, add each pair of numbers in the set. Then check if each sum also belongs to F. Note that the sum of 2 and 8 is 10, which is not an element of F. Thus, F is not closed under addition.

Key Concept

Commutative Property

Let a and b be real numbers.

Axiom 3. *Commutative Property of Addition*

$$a + b = b + a$$

Axiom 4. *Commutative Property of Multiplication*

$$a \cdot b = b \cdot a$$

Note that by the Commutative Property, rearranging the addends does not affect the sum. Similarly, rearranging the factors does not affect the product.

EXAMPLE 2

The Philippines is recognized as the world's center of marine biodiversity. Among its marine reservoirs are beautiful coral reefs. These coral reefs serve as shelter to more than 900 fish species and 400 coral species in the country. Using the given numbers, show that addition and multiplication are commutative.

Solutions:

a. $400 + 900 \stackrel{?}{=} 900 + 400$

$1,300 = 1,300$

The two numbers can be added in any order without affecting the sum.

b. $400 \bullet 900 \stackrel{?}{=} 900 \bullet 400$

$360,000 = 360,000$

The two numbers can be multiplied in any order without affecting the product.

Key Concept

Associative Property

Let *a*, *b*, and *c* be real numbers.

Axiom 5. *Associative Property of Addition*

$$a + (b + c) = (a + b) + c$$

Axiom 6. *Associative Property of Multiplication*

$$a \bullet (b \bullet c) = (a \bullet b) \bullet c$$

The next example illustrates the Associative Property.

EXAMPLE 3

Show that each equation is correct.

a. $4 + (7 + 1) = (4 + 7) + 1$

b. $2 \bullet (3 \bullet 4) = (2 \bullet 3) \bullet 4$

Solutions:

a. $4 + (7 + 1) \stackrel{?}{=} (4 + 7) + 1$

$4 + 8 \stackrel{?}{=} 11 + 1$

$12 = 12$

When adding numbers, regrouping can be done without affecting the sum.

b. $2 \bullet (3 \bullet 4) \stackrel{?}{=} (2 \bullet 3) \bullet 4$

$2 \bullet 12 \stackrel{?}{=} 6 \bullet 4$

$24 = 24$

When multiplying numbers, regrouping can be done without affecting the product.

You can apply the Associative and the Commutative Properties when doing mental computations. When adding three or more numbers, you may first combine the addends that add up to 10 or to any multiple of 10 to make computations easier. Look at the following illustration:

$$\underbrace{\boxed{7} + 4 + \boxed{3}}_{\boxed{7}+\boxed{3}\,=\,10} = 14$$

When multiplying three or more numbers, you may first multiply factors that give a product of 10 or any multiple of 10. Refer to the following illustration:

$$\underbrace{\textcircled{5} \bullet 7 \bullet \textcircled{2}}_{\textcircled{5}\bullet\textcircled{2}\,=\,10} = 70$$

Other properties of real numbers include the existence of the identity element and the inverse element. Study the following axioms:

Key Concept

Existence of an Identity Element

Let a be any real number.

Axiom 7. *Additive Identity Property*

There exists a real number 0, called the *additive identity element*, such that

$$a + 0 = a \text{ and } 0 + a = a.$$

Axiom 8. *Multiplicative Identity Property*

There exists a real number 1, called the *multiplicative identity element*, such that

$$a \bullet 1 = a \text{ and } 1 \bullet a = a.$$

EXAMPLE 4

Find the number that will complete each equation.

a. 4 + ____ = 4

b. 19 • ____ = 19

Solutions:

a. $4 + 0 = 4$ Adding 0 to a number gives a sum equal to that number.

b. $19 \bullet 1 = 19$ Multiplying a number by 1 gives a product equal to that number.

Key Concept

Existence of an Inverse Element

Axiom 9. *Additive Inverse Property*

For any real number a, there exists a real number $-a$, called the *additive inverse* of a, such that

$$a + (-a) = 0 \text{ and } (-a) + a = 0.$$

Axiom 10. *Multiplicative Inverse Property*

For any nonzero real number a, there exists a multiplicative inverse $\frac{1}{a}$, such that

$$a \bullet \frac{1}{a} = 1 \text{ and } \frac{1}{a} \bullet a = 1.$$

EXAMPLE 5

Find the number that will complete each equation.

a. 7 + ____ = 0

b. 4 • ____ = 1

Solutions:

a. $7 + (-7) = 0$ The additive inverse of a real number is obtained by changing the sign of the number. Hence, –7 is the additive inverse of 7. Equivalently, 7 is the additive inverse of –7.

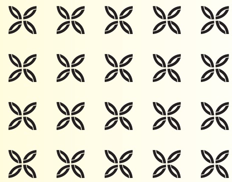
b. $4 \bullet \frac{1}{4} = 1$ The multiplicative inverse of a nonzero real number is its reciprocal. Therefore, the multiplicative inverse of 4 is $\frac{1}{4}$. Conversely, the multiplicative inverse of $\frac{1}{4}$ is 4.

EXAMPLE 6

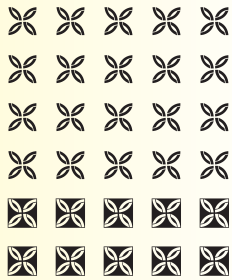
The top four eggplant-producing provinces in the country are Pangasinan, Quezon, Cebu, and Misamis Oriental. Suppose that you planted 4 rows of eggplant seedlings, where each row contains 5 seedlings. Then you added 2 more rows with the same number of eggplant seedlings per row. How many eggplant seedlings did you plant in all?

Solution:

Let  represent an eggplant seedling that you planted initially. You already had 4 rows, where each row contains 5 seedlings, as shown here.



Now let  represent each additional eggplant seedling you planted. Since you added 2 rows, you will have the following number of eggplant seedlings:



There are two ways to solve the problem numerically.

Method A:

Step 1. Identify the total number of rows. If you added 2 more rows, then there will be $4 + 2 = 6$ rows of seedlings.

Step 2. Find the total number of seedlings by multiplying the number of rows by the number of seedlings per row. Hence, there are a total of $5 \cdot 6 = 30$ seedlings.

The solution based on this method may be summarized as follows:

$$\begin{aligned} 5 \cdot (4 + 2) &= 5 \cdot 6 \\ &= 30 \end{aligned}$$

Method B:

Step 1. Find the original number of seedlings that you planted. There are 4 rows, where each row contains 5 eggplant seedlings. Thus, there are $5 \cdot 4 = 20$ seedlings.

Step 2. Find the number of seedlings you added. You added 2 rows, with 5 eggplant seedlings in every row. This means that you added $5 \cdot 2 = 10$ seedlings.

Step 3. Find the total number of seedlings. Add the results in steps 1 and 2. So you will have $20 + 10 = 30$ seedlings.

The solution based on this method is summarized as follows:

$$5 \cdot 4 + 5 \cdot 2 = 20 + 10$$
$$= 30$$

Note that you obtained the same answer as in the previous method. Hence, you planted a total of 30 seedlings.

In the previous example, notice that $5 \cdot (4 + 2) = 5 \cdot 4 + 5 \cdot 2$. This equation illustrates the *Distributive Property of Multiplication Over Addition.*

Key Concept

Distributive Property of Multiplication Over Addition

Let *a*, *b,* and *c* be real numbers.

Axiom 11. $a \cdot (b + c) = a \cdot b + a \cdot c$

This property states that when multiplying a number by a sum of two or more numbers, you can multiply first the number by each of the addends. Then add the resulting products. You can use this property when multiplying numbers, particularly in cases when rewriting one of the factors as a sum could make the computation easier. Study the next example.

EXAMPLE 7

Perform the indicated operations. Apply the Distributive Property to make your computations easier.

a. $120 \cdot 64 + 120 \cdot 36$

b. $80 \cdot 104$

Solutions:

a. $$120 \cdot 64 + 120 \cdot 36 = 120 \cdot (64 + 36)$$
$$= 120 \cdot 100$$
$$= 12{,}000$$

Notice that 64 and 36 are added first to get 100, which can be easily multiplied by 120.

b. To find the product of $80 \cdot 104$, you may think of 104 as $100 + 4$ since it is easier to multiply a number by multiples of 10.

$$80 \cdot 104 = 80 \cdot (100 + 4)$$
$$= 80 \cdot 100 + 80 \cdot 4$$
$$= 8{,}000 + 320$$
$$= 8{,}320$$

You have seen that applying the properties of real numbers could help make your computations easier. You can also use these basic properties to derive or prove other properties of real numbers.

Key Concept

Properties that can be proven using axioms are called **theorems**.

There are no absolute rules in proving theorems. To prove a theorem, you may write a sequence of statements, each of which follows directly from the preceding statement. Each statement must be supported by a reason, which can be an axiom, a definition, or a previously proven theorem. A clear understanding of the properties of real numbers would be a big help when proving theorems. Study the next example.

Example 8

Prove that $c \bullet (a + b) = c \bullet b + c \bullet a$, for any real numbers a, b, and c.

Solution:

To prove the given property, manipulate the expression on one side of the equation to yield the expression on the other side. In this case, you may start with the left side of the equation. Then apply the properties of real numbers to end up with the expression on the right side. You may use a two-column proof. The first column gives the statements, while the second column gives the corresponding reasons. So you will have the following:

Statement	Reason
$c \bullet (a + b) = c \bullet (b + a)$	Commutative Property of Addition
$= c \bullet b + c \bullet a$	Distributive Property

Since you already obtained the expression on the right side of the equation, you have shown that the property is true.

Have you noticed that the previously presented properties of real numbers involve only addition and multiplication? Can you explain why?

In earlier grade levels, you have learned that subtraction is the inverse of addition. You have also learned that division is the inverse of multiplication. Study how subtraction and division are defined based on their inverses.

Key Concept

- **Subtraction**

  If a, b, and c are real numbers, the difference $a - b$ is the number c such that $c + b = a$.

- **Division**

  If a, b, and c are real numbers, the quotient $\frac{a}{b}$, where $b \neq 0$, is the number c such that $c \cdot b = a$.

EXAMPLE 9

Prove each equation using the definition of subtraction or division.

a. $100 - 47 = 53$

b. $264 \div 11 = 24$

Solutions:

a. $100 - 47 = 53$, since $47 + 53 = 100$.

b. $264 \div 11 = 24$, since $24 \cdot 11 = 264$.

Which of the presented axioms hold true for subtraction? For division? Verify your answers by giving examples using different pairs of numbers.

PRACTICE 2-1

Concepts and Skills

Write *True* if the statement is true; otherwise, write *False*.

1. 0 is a natural number.
2. All numbers have a multiplicative inverse.
3. *No* natural number is an irrational number.
4. The additive identity element of the set of integers is 0.
5. A rational number is a quotient of any two integers.
6. The product of a number and its additive inverse is 1.
7. The set of counting numbers is *not* closed under division.
8. Subtraction is associative, but it is *not* commutative.
9. Subtracting the additive identity element from any number results in the given number.
10. The union of the sets of integers and rational numbers is the set of real numbers.

Determine the set(s) where each number belongs to. Put a check mark on the correct column. You may use a calculator when necessary.

	Number	$\mathbb{N}$	$\mathbb{W}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Q}'$
11.	0					
12.	2π					
13.	$\frac{51}{17}$					
14.	$\sqrt{2}$					
15.	$\sqrt{16}$					
16.	3.14					
17.	$-13\frac{1}{7}$					
18.	$0.34\overline{5}$					
19.	–3.8333...					
20.	0.12321213...					

Write *True* if the statement is true; otherwise, write *False*.

21. $\mathbb{N} \subset \mathbb{W}$

22. $\mathbb{Z} \subset \mathbb{Q}$

23. $\mathbb{Q}' \subset \mathbb{Q}$

24. $\mathbb{Q}' = \mathbb{R}$

25. $\mathbb{R} - \mathbb{Q}' = \mathbb{Q}$

26. $\mathbb{Q} \cap \mathbb{N} = \mathbb{Z}$

27. $\mathbb{Z} \cap \mathbb{W} = \mathbb{N}$

28. $\mathbb{Q} \cap \mathbb{Q}' = \varnothing$

29. $\mathbb{N} \cap \mathbb{W} = \varnothing$

30. $\mathbb{W} - \mathbb{N} = \{0\}$

Write the property of real numbers that each equation shows.

31. $\frac{1}{2} \cdot 2 = 1$

32. $\frac{1}{2} + 0 = \frac{1}{2}$

33. $2 \cdot x \cdot y = 2 \cdot y \cdot x$

34. $(-1) + 1 = 0$

35. $z \cdot (x \cdot y) = (x \cdot y) \cdot z$

36. $x + y = y + x$

37. $2 \cdot (x \cdot y) = (2 \cdot x) \cdot y$

38. $x \cdot y + 1 = 1 + x \cdot y$

39. $(-1) + 1 = 1 + (-1)$

40. $(-4) \cdot 2 = 2 \cdot (-4)$

41. $(a + b)(c + d) = (c + d)(a + b)$

42. $(a + b)(c + d) = a(c + d) + b(c + d)$

For items **43** to **45**, complete the two-column proof.

For any real numbers x, a, b, and c, prove that:

$$x \cdot (a + b + c) = x \cdot a + x \cdot b + x \cdot c.$$

Statement	Reason
$x \cdot (a + b + c) = x \cdot [(a + b) + c]$	**43.**
$= x \cdot (a + b) + x \cdot c$	**44.**
$= x \cdot a + x \cdot b + x \cdot c$	**45.**

Applications

Answer each item completely.

46. Jaymie took an examination that is divided into three parts. She got 13 correct answers in the first part, 25 in the second part, and 27 in the third part. Apply the properties of real numbers to find her total score in the exam. Solve for the answer mentally.

47. An elevator can carry a maximum of 9 persons with an average weight of 50 kg each. Suppose that a box of items weighs 9 kg. How many boxes of items can the elevator carry? Explain your answer.

48. Bob teaches 4 hours a day, 4 days a week. His hourly rate is ₱150. How many weeks should he work to earn at least ₱7,000?

For items **49** to **51**, consider the following situation:

Janice buys 12 cupcakes that cost ₱8.50 each. She also buys 8 muffins that cost ₱11 each. Instead of doing the conventional multiplication process, she computes for the total price of the cupcakes and muffins as shown here. Write the properties of real numbers that she used.

$8.5(10 + 2) + 11(10 - 2)$	$= 85 + 17 + 110 - 22$	**49.** ____________________
	$= 85 + 110 + 22 - 5 - 22$	Commutative Property of Addition
	$= 195 - 5 + 22 + (-22)$	Commutative Property of Addition
	$= 190 + 0$	**50.** ____________________
	$= 190$	**51.** ____________________

Enrichment Exercises

Answer each item completely.

52. Is any integer a rational number? Justify your answer.

53. Is the set of odd whole numbers closed under addition? Why or why not?

54. Consider the following equation:

$$(3 \bullet 4) + (5 \bullet 6) = 3 \bullet 4 + 5 \bullet 6.$$

Can you say that $3 \bullet (4 + 5) = 3 \bullet 4 + 5$? Justify your answer.

55. Can you find values for a, b, and c that will make the given equation true. Explain your answer?

$$ab - ac = a - bc$$

LESSON 2-2

Integers

Learning Objectives

At the end of the lesson, you should be able to:

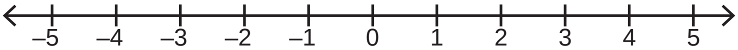
- define integers
- locate integers on a number line
- compare and order integers
- find the absolute value of an integer or expression involving integers
- perform operations on integers

PROBE AND LEARN

The Marianas Trench in the Pacific Ocean is considered as the deepest part of the ocean. It has a depth of about 10,994 m. The Philippines' deepest point, which is also the world's second deepest spot, is the Mindanao Trench. It is about 10,540 m below sea level. Like ocean depths, heights of mountains are also measured from sea level. Mount Apo, the highest mountain in the Philippines, has a height of 2,954 m. Mount Everest, the highest mountain in the world, has a height of 8,848 m.

Ocean depths are measured using sound navigation and ranging (sonar). This is done by sending sound waves from a ship towards the ocean bottom. The time it takes for sound waves to travel to the seafloor and back to the ship is accurately measured and recorded. On the other hand, a surveying altimeter is used to measure the height of mountains. It directly records the altitude of a mountain when it is brought at the top of the mountain. Such measures of heights and depths are good examples of the existence of positive and negative values.

Look at the following number line. Describe the labeled numbers on the left of 0. How do these numbers compare with the set of natural numbers?



Set of Integers

Key Concept

The set of **integers**, which is denoted as $\mathbb{Z}$, is composed of the natural numbers, their additive inverses, and 0.

The set of integers may be described as follows:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Note that the set of integers is the union of three disjoint sets, namely, the set of positive integers $\mathbb{Z}^+ = \{1, 2, 3, 4, 5, \ldots\}$, the set containing 0, and the set of negative integers $\mathbb{Z}^- = \{\ldots, -5, -4, -3, -2, -1\}$. It may be represented as follows:

$$\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+.$$

Note that the set of positive integers is equal to the set of natural numbers.

Mountain heights and ocean depths are measured with respect to sea level, which may be referred to as the reference point. Thus, the sea level may be represented by 0. The phrases *above sea level* and *below sea level* indicate values that have opposite signs. Distances below sea level may be represented by negative numbers. Distances above sea level may be represented by positive numbers. Hence, the given ocean depths and mountain heights may be written using signed numbers as follows:

Place		Distance from Sea Level (in meters)
Above Sea Level	Mount Everest	8,848
	Mount Apo	2,954
Below Sea Level	Mindanao Trench	–10,540
	Marianas Trench	–10,994

Positive numbers may be written without a plus sign (+). Thus, +4 is the same as 4. Negative numbers are written with a minus sign (–) before them. For example, negative 4 is written as –4. Zero is neither positive nor negative, so it is written without a + or – sign.

EXAMPLE 1

Express each numerical value as an integer.

a. The 2018 General Wholesale Price Index (GWPI) in the Philippines dropped by about 2% during a certain month.

b. The 2013 functional literacy rate of the country grew by about 8 percentage points from 2008.

c. Fuelwood consumption in rural areas declined by 6%.

Answers:

a. –2 The word *dropped* suggests a negative integer.

b. 8 The word *grew* indicates a positive integer.

c. –6 *Why?*

EXAMPLE 2

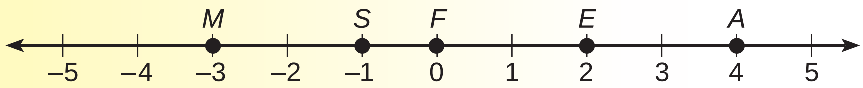
The following table shows the differences in the grades of a student from one grading period to the next grading period:

Subject	Difference
English	+2
Filipino	0
Science	–1
Math	–3
Araling Panlipunan	+4

Plot each integer on a number line. Label the points properly.

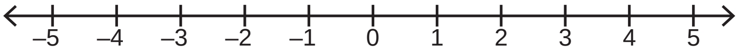
Solutions:

Use the following letters to label the points: *E* for English, *F* for Filipino, *S* for Science, *M* for Math, and *A* for Araling Panlipunan. So you will have:



Comparing and Ordering Integers

Look at the following number line:



A number line can help you in comparing and ordering integers. Note that the values of integers are increasing from left to right on the number line. This means that the integer on the right of a given integer is greater, and any other integer on its left is lesser in value. Also, remember that any negative integer is less than 0, while any positive integer is greater than 0. Can you say that any positive integer is greater than any negative integer?

Example 3

Refer to the given table in the previous example. In which subject does the student have the greatest increase in grades? How about the greatest decrease?

Solution:

You can use the number line you made in the previous example to arrange the integers from least to greatest. Consider that the integers are written in order from left to right on the number line.

M	*S*	*F*	*E*	*A*
–3	–1	0	2	4

Therefore, the greatest increase is 4, which is in Araling Panlipunan. The greatest decrease is –3, which is in Math.

To compare two integers, the following symbols may be used:

Symbol	Meaning
=	is equal to
>	is greater than
<	is less than

Example 4

Compare each pair of integers using >, <, or =.

a. 1 _____ 5 b. –1 _____ –2 c. –4 _____ 0

Solutions:

a. 1 < 5 1 is located on the left side of 5 on the number line.

b. –1 > –2 –1 is located on the right side of –2 on the number line.

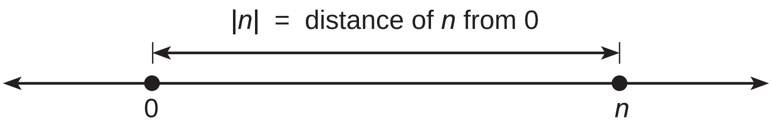
c. –4 < 0 *Why?*

Absolute Value

Key Concept

The distance of a number from 0 on a number line is called the **absolute value** of the number. The absolute value of an integer n is written as $|n|$.

Note that since the absolute value refers to a distance, it is always nonnegative.



Let n be any integer.

- $|n| = n$ if n is a positive integer.
- $|n| = 0$ if n equals 0.
- $|n| = -n$ if n is a negative integer.

Do you know?

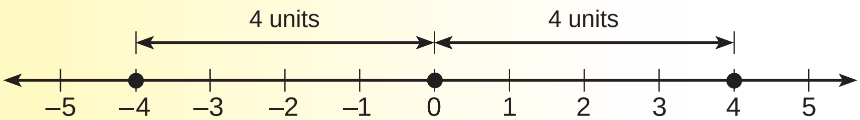
You can think of the absolute value of a number as the number itself, but without the sign. For example, $|-47| = 47$.

EXAMPLE 5

Find $|4|$, $|-4|$, and $|0|$.

Solutions:

First, locate each integer on the number line. Then count how many units away each integer is from 0.



Since 4 is 4 units away from 0, then $|4| = 4$.

Since -4 is 4 units away from 0, then $|-4| = 4$.

Notice that $|0| = 0$. *Why?*

Operations on Integers

Like with whole numbers, performing operations on integers follows certain rules. Your knowledge of the concept of absolute value will be of great help in understanding these rules. Study how addition, subtraction, multiplication, and division on integers are performed.

Addition of Integers

You can find *sari-sari* stores anywhere in the Philippines. Amy owns a sari-sari store. Every week, she lists down her profits and losses. The following is a summary of her record for four weeks:

Week 1	a profit of ₱1,426
Week 2	a loss of ₱292
Week 3	a loss of ₱585
Week 4	a profit of ₱320

A sari-sari store is a small convenience store where you can buy items by piece or by small packages.

Did Amy earn or lose money in those four weeks? How much did she earn or lose?

Profits imply positive numbers, while losses imply negative numbers. Hence, each amount in the table may be represented by an integer. You can solve the problem by finding the sum of the integers. Let N represent the sum. So you will have the following equation:

$$N = 1{,}426 + (-292) + (-585) + 320$$

How do you solve for N? First, identify the rules in adding integers. Remember that positive integers are counting numbers. This means that the sum of two positive integers is also positive.

How about the sum of two negative integers? Or the sum of a positive integer and a negative integer? You could better understand how to add integers if you use a number line. Study how it could be done.

First, locate one of the addends on the number line. Suppose that n is the other addend.

If n is positive, move $|n|$ units to the right of the first addend.

If n is negative, move $|n|$ units to the left of the first addend.

Study the next example.

EXAMPLE 6

Find each sum.

a. $(-4) + 7$

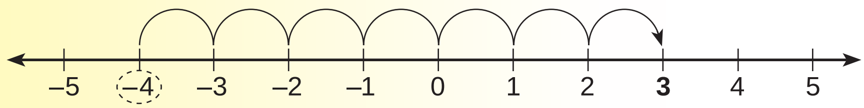
b. $(-4) + (-7)$

Solutions:

a. (–4) + 7

❶ Start at –4.

❷ Move 7 units to the right.

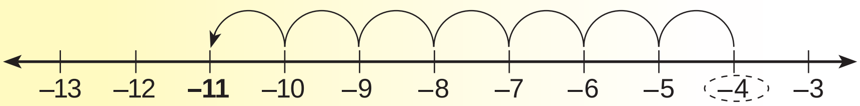


Thus, (–4) + 7 = 3.

b. (−4) + (−7)

❶ Start at –4.

❷ Move 7 units to the left.



Thus, (–4) + (–7) = −11.

In item b of the previous example, you found out that (–4) + (–7) = −11. This means that the sum of two negative integers is also negative. Use the following rule when adding integers with like signs:

Key Concept

When adding integers with like signs, add their absolute values. Then attach the common sign to the sum.

Another way to illustrate addition of integers is by using counters. Counters are useful in showing the rule in adding integers with different signs.

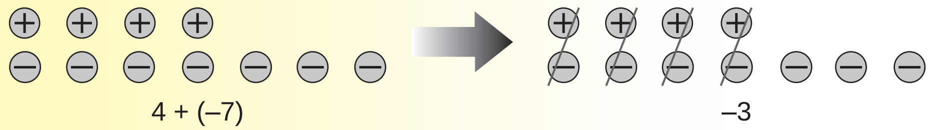
Let ⊕ represent +1. Let ⊖ represent −1. Note that a ⊕ and a ⊖ make a *zero pair*. Since a zero pair has a value of 0, you can remove zero pairs without affecting the resulting answer. Study next example.

EXAMPLE 7

Find the sum of 4 + (–7) using counters.

Solution:

Represent each integer using counters. Then remove any zero pair, if there are any. The number that represents the remaining counters is the sum.



Hence, 4 + (–7) = –3.

In examples 6 and 7, you found out that (–4) + 7 = 3 and 4 + (–7) = –3. Notice that the sum in each of the given equations is the difference of the absolute values of the addends, and the sign of the sum is the sign of the addend with the greater absolute value. This shows the next rule.

Key Concept

When adding integers with unlike signs, subtract their absolute values. Then attach the sign of the integer with the greater absolute value to the sum.

The rule means that the sum of a positive integer and a negative integer may be either negative or positive, depending on the sign of the addend that has the greater absolute value. Can their sum be equal to 0?

EXAMPLE 8

Find the sum of (−9) + 4.

Solution:

$\lvert -9 \rvert - \lvert 4 \rvert = 5$	Find the difference of the absolute values of the integers.
$(-9) + 4 = -5$	Attach to the sum the sign of the integer with the greater absolute value. Since –9 has the greater absolute value, the sign of the sum is negative.

You can now solve the problem posed at the start of the section on adding integers. The problem asked for Amy's gain or loss from her sari-sari store in four weeks. To solve it, you have to find the value of *N* in the following equation:

$$N = 1{,}426 + (-292) + (-585) + 320.$$

Note that the addends in the equation include positive and negative integers. Recall that addition of real numbers is commutative and associative. Thus, you can group together and add integers of the same sign. Then add the resulting sums. So you will have:

$$N = (1{,}426 + 320) + [(-292) + (-585)]$$
$$= 1{,}746 + (-877)$$
$$= 869$$

The sum is positive. This means that Amy has a profit of ₱869 in those four weeks.

Subtraction of Integers

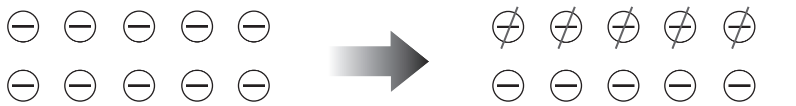
Different cooperatives help people through their loan programs. Suppose that Susan borrows ₱10,000 from a credit cooperative to start a small business. After a few months, she pays ₱5,000 back. How much does she still owe the cooperative, assuming no interest will be applied?

An amount of borrowed money implies a negative number. In this problem, the amount Susan borrows can be represented by –10,000. The act of paying back may be expressed by subtracting a negative number. Since Susan pays ₱5,000 back, you have to subtract –5,000 from –10,000.

Let N be the amount that Susan still has to pay. Hence,

$$N = (-10{,}000) - (-5{,}000).$$

To find N, you need to subtract integers. Like in adding integers, you may use counters in subtracting integers. For this problem, let ⊖ represent ₱1,000.



–10,000   –10,000 – (–5,000) = –5,000

Therefore, Susan still owes the cooperative ₱5,000.

You may also use a number line to subtract integers. First, locate the minuend on the number line. Suppose that n is the subtrahend.

- If n is negative, move $|n|$ units to the right of the minuend.
- If n is positive, move $|n|$ units to the left of the minuend.

Consider the next examples.

EXAMPLE 9

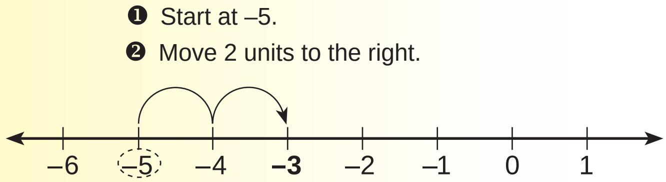
Find each difference.

a. (–5) – (–2)

b. (–5) – 2

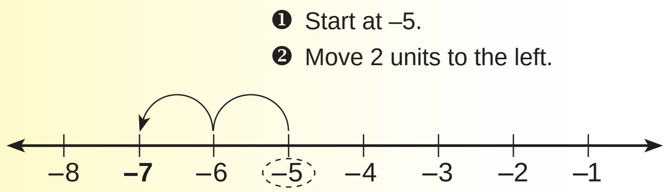
Solutions:

a. (–5) – (–2)



Thus, (–5) – (–2) = –3.

b. (–5) – 2



Thus, (–5) – 2 = –7.

In item a of the previous example, notice that the number line also shows (–5) + 2. Thus,

$$(-5) - (-2) = (-5) + 2.$$

In item b, the number line also shows (–5) + (–2). Thus,

$$(-5) - 2 = (-5) + (-2).$$

These illustrate that you can rewrite a subtraction problem into an addition problem. This is done by changing the subtrahend to its additive inverse, and then changing the minus sign to a plus sign. Therefore, the rule for subtracting integers may be stated as follows:

Key Concept

When subtracting integers, change the subtrahend to its additive inverse, and change the operation to addition. Then follow the rules in adding integers.

Study the following illustration:

$(-5) - (-2)$

❷ Change to addition. ↓ ↓ ❶ Get the additive inverse.

$(-5) + 2 = -3$ ❸ Add the integers.

Example 10

Subtract.

a. $7 - (-9)$

b. $6 - 18$

c. $-3 - 12$

d. $-9 - (-4)$

Solutions:

a. $7 - (-9) = 7 + 9 = 16$

b. $6 - 18 = 6 + (-18) = -12$

c. $-3 - 12 = -3 + (-12) = -15$

d. $-9 - (-4) = -9 + 4 = -5$

Key Concept

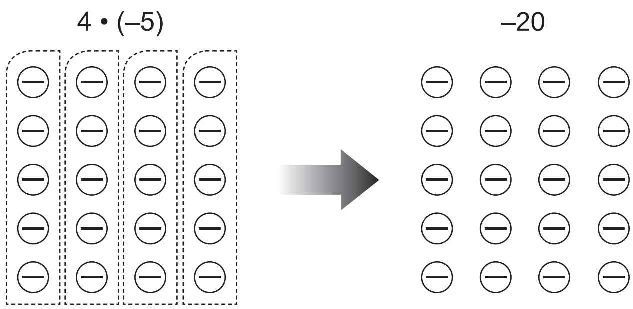
In general, if a and b are real numbers, the following equivalence relations exist:

- $a + (-b) = a - b$
- $a - (+b) = a - b$
- $a - (-b) = a + b$

Multiplication of Integers

Suppose that a bus travels at an average rate of 80 kph. It decreases its speed by 5 kph after every hour. What is its speed after 4 hours?

To find the speed of the bus after 4 hours, determine by how much the bus has decreased its speed after 4 hours. A decrease of 5 kph in speed may be represented by –5. Now determine how many times the bus decreases its speed in 4 hours. Since its speed decreases every hour, the bus decreases its speed 4 times. To find the total decrease in speed after 4 hours, get the product of 4 • (–5). You may solve this using counters. Let ⊖ represent –1.



Hence, the speed of the bus after 4 hours is given by:

$$80 + (-20) = 60 \text{ kph.}$$

You may also use counters to verify the following rules in multiplying integers:

Key Concept

When multiplying integers, get the product of their absolute values. Then apply the following rules for the sign of the product:

- The product of two integers with like signs is positive.
- The product of two integers with unlike signs is negative.

EXAMPLE 11

Multiply.

a. $(-15)(-8)$ b. $9 \cdot (-12)$

Solutions:

a. $(-15)(-8) = 120$ The two factors have like signs. Hence, their product is positive.

b. $9 \cdot (-12) = -108$ The two factors have unlike signs. Hence, their product is negative.

When multiplying three or more integers, get the product of their absolute values. Then determine the sign of the product by counting the negative factors. An odd number of negative factors gives a negative product. An even number of negative factors gives a positive product. Can you explain why?

EXAMPLE 12

Find the product of $(-5)(-2)(-3)$.

Solution:

$5 \cdot 2 \cdot 3 = 30$ Multiply the absolute values of the given integers.

$(-5)(-2)(-3) = -30$ The product is negative since there are three negative factors.

Division of Integers

Baguio City is considered as the summer capital of the Philippines due to its relatively low temperature. Its temperatures range from 15°C to 26°C. On a particular day in December 2018, the temperature reading in Baguio at 3 p.m. is 21°C. Suppose that the temperature decreases by 2°C every hour. At what time will the temperature reading be 15°C?

A decrease in temperature implies a negative number. If the temperature drops from 21°C to 15°C, then the total decrease in temperature may be represented by 15 – 21 = –6, or −6°C. After how many hours will the temperature be 15°C? Recall that the temperature decreases by 2°C every hour. A decrease of 2°C may be represented by −2.

To solve this problem, you have to divide integers; that is, divide –6 by –2. Since division is the inverse of multiplication, the same rules of signs apply. You can verify these rules using counters.

Key Concept

When dividing integers, get the quotient of their absolute values. Then apply the following rules for the sign of the quotient:

- The quotient of two integers with like signs is positive.
- The quotient of two integers with unlike signs is negative.

Therefore, the solution to the previous problem is:

$$(-6) \div (-2) = 3.$$

This means that it will take 3 hours for the temperature to decrease from 21°C to 15°C. Hence, the temperature reading will be 15°C at 6 p.m.

EXAMPLE 13

Divide.

a. $(-15) \div (-3)$

b. $132 \div (-12)$

Solutions:

a. $(-15) \div (-3) = 5$ The two integers have like signs. Hence, their quotient is positive.

b. $132 \div (-12) = -11$ The two integers have unlike signs. Hence, their quotient is negative.

Practice 2-2

Concepts and Skills

Express each numerical value using integers.

1. Today is warmer than yesterday by 5°C.
2. Joy lost 2 kg after months of jogging.
3. John deposited ₱20,000 in his savings account.
4. Mary needs to run 5 more meters to reach the finish line.
5. The mystical Mount Banahaw is 2,169 m above sea level.

Write *True* if the statement is true; otherwise, write *False*.

6. The additive inverse of 4 is –4.
7. The quotient of any two integers is an integer.
8. The absolute value of any integer is positive.
9. The product of three negative integers is negative.
10. The sum of two integers is greater than each of the addends.
11. The quotient of a positive number and its additive inverse is –1.
12. If a is a negative number and b is a positive number, then $a \cdot (-b)$ is positive.
13. If the product of two integers is positive, then one of the integers could be negative.

Perform the indicated operations.

14. 36 + 9
15. 36 + (–9)
16. (–36) + 9
17. (–36) + (–9)
18. 36 – 9
19. 36 – (–9)
20. (–36) – 9
21. (–36) – (–9)
22. 36 • 9
23. 36 • (–9)
24. (–36) • 9
25. (–36) • (–9)
26. 36 ÷ 9
27. 36 ÷ (–9)
28. (–36) ÷ 9
29. (–36) ÷ (–9)
30. 5 • [10 – (–2)]
31. (–9) + 10 + (–11)
32. 100 – (–25) – 50
33. 20 • (–25) ÷ (–5)
34. 15 • 30 + (–10)
35. 30 ÷ (–2) – (–15)
36. 3 • [17 – (–13)] ÷ (–2)

Write the letter of the statement that best describes each pair of quantities in the two columns.

A. The quantity in column 1 is greater than the quantity in column 2.

B. The quantity in column 2 is greater than the quantity in column 1.

C. The two quantities are equal.

D. The two quantities *cannot* be compared.

	Column 1	Column 2
37.	$\lvert 0 \rvert$	$0 \bullet (-5)$
38.	$-(-1)$	$\lvert -3 \rvert - 2$
39.	$5 \bullet (-4)$	$(-80) \div (-4)$
40.	$15 - (-5)$	$-15 + 5$
41.	n	$\lvert n \rvert$

Find the next two terms in each sequence.

42. –16, –11, –6, –1, 4, ______, ______

43. 729, –243, 81, –27, 9, ______, ______

44. 1, –1, 2, –2, 4, –4, 8, –8, ______, ______

45. –2, 8, 2, 5, 6, 2, 10, –1, ______, ______

46. 0, 0, 1, –1, 2, –4, 3, –9, ______, ______

Applications

For items **47** to **53**, refer to the following table that shows the number of enrollees in a certain high school for the school year (SY) 2009–2010 and SY 2010–2011.

Grade Level	SY 2009–2010	SY 2010–2011
Grade 7	258	342
Grade 8	310	295
Grade 9	365	370
Grade 10	375	360

47. What is the change in the number of enrollees in each grade level from SY 2009–2010 to SY 2010–2011? Express each change as an integer.

Grade Level	Change in the Number of Enrollees
Grade 7	
Grade 8	
Grade 9	
Grade 10	

48. Plot the data you obtained in the previous item on a number line. Use the grade levels to label the points.

49. In which grade level is the increase in the number of enrollees greatest?

50. In which grade level is the decrease in the number of enrollees greatest?

51. Which level has a greater change in the number of enrollees from SY 2009–2010 to SY 2010–2011: grade 8 or grade 9? By how many students?

52. What is the change in the total number of enrollees from SY 2009–2010 to SY 2010–2011?

53. What is the average change in the number of enrollees from SY 2009–2010 to SY 2010–2011?

Answer each item completely.

54. What is the distance between the lowest point and the highest point in the Philippines? Refer to the data given at the start of this lesson.

55. In a 20-item test, a student receives 5 points for each question answered correctly. Two points are subtracted from the total score for each question answered incorrectly. One point is subtracted for each unanswered question. If a student has 10 correct answers and 6 incorrect answers, what is his total score in the test?

56. The Philippine national debt rose to about 4 trillion pesos in 2008. Suppose that there are approximately 75 million Filipinos. What is the average debt of each Filipino?

57. Nora records the temperatures in her place at different times. At midnight, the temperature is 15°C. By 4 a.m., it drops by 6°C. It then begins to rise at a rate of 2°C every hour until noon, when the rate of increase becomes 3°C every hour. At 4 p.m., the temperature begins to decrease by 2°C every hour. What is the temperature at 6 p.m.?

Enrichment Exercises

Answer each item completely.

58. If $|x| < |y|$, is it true that $x < y$? Justify your answer.

59. Consider the equation $|x| + |y| = |x + y|$. Could x and y have different signs?

60. Suppose that a negative number is subtracted from a positive number. Is the difference positive or negative? Explain your answer.

LESSON 2-3

Rational Numbers

Learning Objectives

At the end of the lesson, you should be able to:

- define rational numbers
- graph rational numbers on a number line
- compare and order rational numbers
- state the density property of rational numbers
- express fractions as decimals, and vice versa
- perform operations involving rational numbers
- simplify complex fractions
- solve word problems involving rational numbers

Probe and Learn

Plants grow new cells in spirals so that the new leaves do not block the older leaves from getting sunlight. Each new cell is formed about 137.5° away from the previous cell. This angle is called the *golden angle*. In geometry, the golden angle is the smaller of the two angles formed by sectioning the circumference of a circle based on the golden ratio.

The number 137.5 is the same as the fraction $\frac{275}{2}$, which is a quotient of two integers. Recall that the set of integers is closed under addition, subtraction, and multiplication, but not under division. For example, the quotient of the integers 275 and 2 is not an integer. Such quotients are part of another subset of the set of real numbers—the set of *rational numbers*.

Set of Rational Numbers

Key Concept

The set of **rational numbers**, denoted by $\mathbb{Q}$, is composed of numbers that can be expressed as a ratio of two integers. In symbols,

$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p \text{ and } q \text{ are integers, and } q \neq 0 \right\}.$$

From the given definition, you can say that fractions are rational numbers. Consider the fraction $\frac{1}{2}$, which is the ratio of the integers 1 and 2. Dividing 1 by 2 gives a decimal quotient of 0.5. Note that any fraction can be expressed either as a *terminating decimal* or a *nonterminating but repeating decimal.*

Fractions and Decimals

A terminating decimal consists of a finite number of decimal digits. Numbers, such as 4, 1.2345, and 0.987654321, that have finite nonzero digits on the right of the decimal point are terminating decimals.

A nonterminating but repeating decimal has an infinitely repeating digit or group of digits. An ellipsis is used to denote that the decimal has an infinite number of digits. For example, 0.33333... has an infinite number of 3s on the right of the decimal point, while the decimal 0.456456456456... has an infinitely repeating group of digits, 456. The repeating digit or group of digits may also be written once with a bar on top, called a *vinculum*. Thus, you can rewrite 0.3333... as $0.\overline{3}$ and 0.456456... as $0.\overline{456}$.

EXAMPLE 1

Write each fraction as a decimal.

a. $\frac{1}{4}$

b. $\frac{1}{3}$

Solutions:

a. Divide 1 by 4.

$$\begin{array}{r} 0.25 \\ 4\overline{)1.00} \\ \underline{-8} \\ 20 \\ \underline{-20} \\ 0 \end{array}$$

Thus, $\frac{1}{4} = 0.25$, which is a terminating decimal.

b. Divide 1 by 3.

$$\begin{array}{r} 0.333... \\ 3\overline{)1.000} \\ \underline{-9} \\ 10 \\ \underline{-9} \\ 10 \\ \underline{-9} \\ 1 \end{array}$$

Therefore, $\frac{1}{3} = 0.\overline{3}$, which is a nonterminating but repeating decimal.

EXAMPLE 2

Some scientific studies say that smiling requires fewer muscles than frowning does—17 muscles to smile and 43 to frown. Write a fraction representing the number of needed muscles to smile as a part of the number of needed muscles to frown. Then express the fraction as a decimal.

Solution:

The number of needed muscles to smile as a part of the number of needed muscles to frown can be represented as $\frac{17}{43}$. To write this fraction as a decimal, divide 17 by 43. The quotient is the nonterminating but repeating decimal 0.3953488.... Observe that the decimal does not seem to show a pattern in its first 7 digits. For a decimal like this, you may round it off to the nearest hundredths as 0.4.

Conversely, you can write any terminating or nonterminating but repeating decimal as a fraction. Study the next examples.

EXAMPLE 3

Express each decimal as a fraction. Write your answer in simplest form.

a. –1.2

b. 0.625

Solutions:

a. –1.2 is read as "negative 1 and 2-tenths."

$$-1.2 = -1\frac{2}{10} \quad \leftarrow \text{The single decimal place in –1.2 corresponds to 1 zero.}$$

$$= -1\frac{1}{5}$$

Hence, $-1.2 = -1\frac{1}{5}$.

b. 0.625 is read as "625-thousandths."

$$0.625 = \frac{625}{1{,}000} \quad \leftarrow \text{The three decimal places in 0.625 correspond to 3 zeros.}$$

$$= \frac{5}{8}$$

Therefore, $0.625 = \frac{5}{8}$.

You can express a rational number as a fraction in infinitely many ways. How do you know if a fraction is already in its simplest form?

A fraction is in simplest form if its numerator and denominator are relatively prime; that is, their greatest common factor (GCF) is 1. For example, the fraction $\frac{5}{7}$ is in simplest form since 5 and 7 are relatively prime.

To simplify fractions, you can apply the *Fundamental Law of Fractions*.

Key Concept

The Fundamental Law of Fractions

If a, b, and c are real numbers, $b \neq 0$ and $c \neq 0$, then $\frac{ac}{bc} = \frac{a}{b}$.

This property means that if the numerator and the denominator of a fraction $\left(\frac{ac}{bc}\right)$ have a common factor (c), you can cancel out the common factor without changing the value of the fraction. Also, when you cancel the GCF of the terms of the fraction, the result is the simplest form of the fraction.

EXAMPLE 4

Simplify $\frac{30}{45}$.

Solution:

First, find the prime factorization of 30 and 45. Then cancel out all of their common factors.

$$\frac{30}{45} = \frac{2 \cdot \overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{5}}}{3 \cdot \underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{5}}} = \frac{2}{3}$$

Since 2 and 3 are relatively prime, $\frac{2}{3}$ is the simplest form of $\frac{30}{45}$.

Now, how do you write a nonterminating but repeating decimal as a fraction?

Note that in this lesson, you will only consider decimals that are made up entirely of repeating digits. You will not deal with decimals that have a combination of nonrepeating digits and repeating digits. In the next example, you will learn a method for converting nonterminating but repeating decimals into fractions.

EXAMPLE 5

Express each decimal as a fraction in simplest form.

a. 0.333...

b. 0.545454...

Solutions:

a. $0.333... = \frac{3}{9}$ ← 3 is the repeating digit. ← Write a 9 for every repeating digit.

$= \frac{1}{3}$ ← Simplify $\frac{3}{9}$.

b. $0.545454... = \frac{54}{99}$ ← 54 is the repeating group of digits. ← Write a 9 for every repeating digit.

$= \frac{6}{11}$ ← Simplify $\frac{54}{99}$.

Knowing how to express rational numbers as fractions or as decimals can help you plot them on a number line. Study the next example.

Example 6

Plot each rational number on a number line.

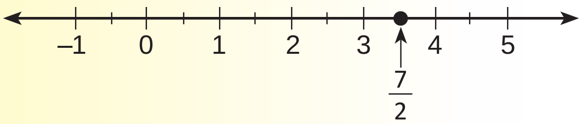
a. $\frac{7}{2}$

b. –2.3

Solutions:

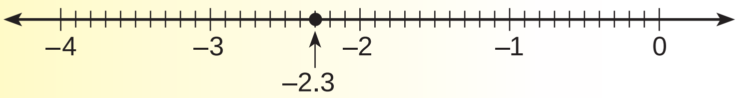
a. You know that $\frac{7}{2} = 3\frac{1}{2}$

To plot $\frac{7}{2}$ on a number line, divide the units of the number line into halves.



b. –2.3

To plot –2.3 on a number line, divide the units of the number line into tenths.



Comparing Rational Numbers

Can you tell how many rational numbers there are between $\frac{1}{4}$ and $\frac{1}{3}$? The answer can be explained using the next property of rational numbers.

Key Concept

Density Property of Rational Numbers

Between any two rational numbers, there is always another rational number.

This means that between any two rational numbers, there are infinitely many rational numbers.

Knowing the positions of rational numbers or their decimal representations on a number line may help you compare fractions. However, you may not need to plot the numbers on a number line to be able to compare them. The following are the rules when comparing rational numbers:

- When comparing two positive rational numbers, compare as you would with whole numbers.

- Any negative rational number is less than any positive rational number.
- When comparing two negative rational numbers, the greater the absolute value of a number, the lesser the number.

EXAMPLE 7

Compare each pair of numbers using <, >, or =.

a. 5.05 ____ 5.005 b. –2.5 ____ 1.5 c. –6.4 ____ –10.2

Solutions:

a. 5.05 > 5.005 The hundredths digit of 5.0**5** is greater than the hundredths digit of 5.0**0**5

b. –2.5 < 1.5 Any negative rational number is less than any positive rational number.

c. –6.4 > –10.2 Since |–6.4| < |–10.2|, then –6.4 > –10.2.

You have learned that to compare fractions, you may need to rename the fractions as similar fractions using the Least Common Denominator (LCD) as their denominator. Recall that the LCD of the fractions is the least common multiple of the denominators. You may then compare the numerators of the similar fractions. The fraction with a greater numerator is the greater fraction.

EXAMPLE 8

Compare $\frac{4}{9}$ and $\frac{5}{12}$.

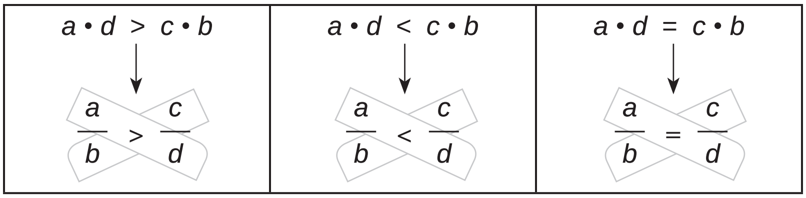
Solution:

The LCD of $\frac{4}{9}$ and $\frac{5}{12}$ is 36. Find fractions equivalent to the given fractions using the LCD as denominator.

$$\frac{4}{9} = \frac{\mathbf{16}}{\mathbf{36}} \;(\times 4) \quad \text{and} \quad \frac{5}{12} = \frac{\mathbf{15}}{\mathbf{36}} \;(\times 3)$$

Compare the numerators of the similar fractions. Since 16 > 15, then $\frac{4}{9} > \frac{5}{12}$.

Another way to compare fractions is by performing cross-multiplication. The following diagrams illustrate how cross-products are used to compare the fractions $\frac{a}{b}$ and $\frac{c}{d}$:



Why can you compare fractions by comparing the resulting cross-products?

Example 9

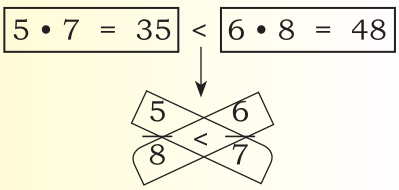
Compare each pair of fractions by getting their cross-products.

a. $\frac{5}{8}$ ——— $\frac{6}{7}$

b. $-\frac{5}{11}$ ——— $-\frac{4}{9}$

Solutions:

a. Compare the cross-products.

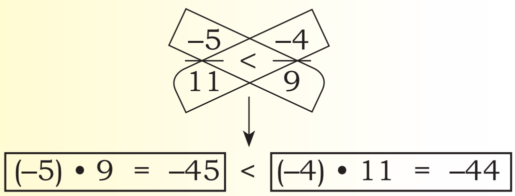


Since 35 < 48, then $\frac{5}{8} < \frac{6}{7}$.

b. Notice that both fractions are negative. You can write a negative rational number in any of the following forms:

$$\frac{-a}{b}, \frac{a}{-b}, \text{ or } -\frac{a}{b}.$$

Hence, you may rewrite $-\frac{5}{11}$ as $\frac{-5}{11}$ and $-\frac{4}{9}$ as $\frac{-4}{9}$.



Since –45 < –44, then $-\frac{5}{11} < -\frac{4}{9}$.

Ordering Rational Numbers

You already know how to compare rational numbers. Now you are ready to order rational numbers from least to greatest, or vice versa. Study the next example.

Example 10

Order the following rational numbers from least to greatest:

$$1.5, \frac{10}{7}, 1\frac{7}{8}, \frac{4}{9}, 0.3, -0.3, -\frac{1}{3}$$

Solution:

Observe that two of the given numbers are negative. Recall that negative numbers are less than positive numbers. So, which is less, -0.3 or $-\frac{1}{3}$? To answer it, write the numbers in the same form, say decimal form. You may express the fraction $-\frac{1}{3}$ as $-0.\overline{3}$. Thus, $-\frac{1}{3} < -0.3$.

Now consider the five remaining numbers: 1.5, $\frac{10}{7}$, $1\frac{7}{8}$, $\frac{4}{9}$, and 0.3. You can see that $0.3 < 1.5$, since both are in decimal form.

Of the three remaining fractions, $\frac{4}{9}$ is the least since it is a proper fraction, which means it is less than 1.

On the other hand, $\frac{10}{7} = 1\frac{3}{7}$, which is less than $1\frac{7}{8}$. Thus, the three fractions may be written from least to greatest as $\frac{4}{9}$, $1\frac{3}{7}$, $1\frac{7}{8}$.

Now consider 0.3 and 1.5. Is 0.3 less than $\frac{4}{9}$? You can express 0.3 as $\frac{3}{10}$, which is less than $\frac{4}{9}$. Since $1.5 = 1\frac{1}{2}$, then $1\frac{3}{7} < 1\frac{1}{2} < 1\frac{7}{8}$. Thus, the following is the correct order of the numbers from least to greatest:

$$-\frac{1}{3}, -0.3, 0.3, \frac{4}{9}, \frac{10}{7}, 1.5, 1\frac{7}{8}$$

Addition and Subtraction of Rational Numbers

Recall that when adding or subtracting decimals, you have to align the decimal points and the digits having the same place value.

EXAMPLE 11

Add or subtract.

a. $10.23 + 5.3 + 27.456$ b. $11.346 + (-3.09)$ c. $-20.5 - 7.12$

Solutions:

a.
$$\begin{array}{r} 10.23 \\ 5.3 \\ +\ 27.456 \\ \hline 42.986 \end{array}$$

b.
$$\begin{array}{r} 11.346 \\ +\ (-3.09) \\ \hline 8.256 \end{array}$$

c.
$$\begin{array}{r} -20.5 \\ -\quad 7.12 \\ \hline \end{array} \rightarrow \begin{array}{r} -20.50 \\ +\ (-7.12) \\ \hline -27.62 \end{array}$$

Recall that when adding or subtracting similar fractions, add or subtract the numerators. Then copy the denominator. Express the sum or difference in simplest form, if necessary.

EXAMPLE 12

Add or subtract.

a. $1\frac{4}{7} + 2\frac{5}{7}$ b. $4\frac{4}{15} - 2\frac{4}{15}$

Solutions:

a. $1\frac{4}{7} + 2\frac{5}{7} = 3\frac{9}{7}$

$= 4\frac{2}{7}$ Rename $\frac{9}{7}$ as $1\frac{2}{7}$.

b.
$$\begin{array}{r} 4\frac{4}{15} \\ -\ 2\frac{14}{15} \\ \hline \end{array} \rightarrow \begin{array}{r} 3\frac{19}{15} \\ -\ 2\frac{14}{15} \\ \hline 1\frac{5}{15} \end{array}$$

Rename $4\frac{4}{15}$ as $3\frac{19}{15}$. Why?

Thus, $4\frac{4}{15} - 2\frac{14}{15} = 1\frac{5}{15}$, which may be written in simplest form as $1\frac{1}{3}$.

To find the sum or the difference of dissimilar fractions, rename the fractions using the LCD. Then follow the rules in adding and subtracting similar fractions.

EXAMPLE 13

Add or subtract.

a. $\frac{3}{10} + \frac{1}{15}$ b. $10\frac{1}{2} - 1\frac{2}{3}$ c. $-\frac{4}{9} - \frac{5}{18}$

Solutions:

a. $\frac{3}{10} + \frac{1}{15} = \frac{9}{30} + \frac{2}{30}$ The LCD of 10 and 15 is 30.

$= \frac{11}{30}$

b. $10\frac{1}{2} - 1\frac{2}{3} = 9\frac{3}{2} - 1\frac{2}{3}$ Rename $10\frac{1}{2}$ as $9\frac{3}{2}$

$= 9\frac{9}{6} - 1\frac{4}{6}$

$= 8\frac{5}{6}$

c. $-\frac{4}{9} - \frac{5}{18} = -\frac{4}{9} + \left(-\frac{5}{18}\right)$ Rewrite it into an addition problem.

$= -\frac{8}{18} + \left(-\frac{5}{18}\right)$

$= -\frac{13}{18}$

Now study the next example that shows how to solve a word problem involving addition and subtraction of fractions.

EXAMPLE 14

Among the Philippines's major agricultural exports are canned pineapples. Jose owns a pineapple farm. He wants to fence his rectangular farm, which has a length of $32\frac{1}{4}$ ft. and a width of $21\frac{1}{3}$ ft. Suppose that fencing wires are sold in rolls of 20 yards. How many rolls of wire should he buy? How many feet of wire will be left over?

Solution:

To find the length of the required wire to fence his rectangular farm, find the perimeter of the farm. Recall that you can solve for the perimeter P of a rectangle by adding the measures of all its sides.

$$P = l + l + w + w, \text{ where } l = \text{length and } w = \text{width}$$

It is given that $l = 32\frac{1}{4}$ ft. and $w = 21\frac{1}{3}$ ft. So you will have the following:

$$\begin{aligned} P &= 32\frac{1}{4} + 32\frac{1}{4} + 21\frac{1}{3} + 21\frac{1}{3} \\ &= 106\frac{7}{6} \\ &= 107\frac{1}{6} \end{aligned}$$

Thus, Jose needs $107\frac{1}{6}$ ft. of fencing wire. Recall that a roll of fencing wire is 20 yards long. Since 1 yd. = 3 ft., then 20 yd. = 60 ft. Thus, he needs 2 rolls of fencing wire, with a total length of 120 ft.

To find how much of the wire will be left over after fencing his farm, subtract $107\frac{1}{6}$ ft. from 120 ft.

$$\begin{aligned} 120 - 107\frac{1}{6} &= 119\frac{6}{6} - 107\frac{1}{6} \\ &= 12\frac{5}{6} \end{aligned}$$

Therefore, $12\frac{5}{6}$ ft. of wire will be left over.

Multiplication of Rational Numbers

Recall that when multiplying decimals, multiply the numbers as you would with whole numbers. To determine the position of the decimal point in the product, count the decimal places in the factors. The total number of decimal places in the factors is equal to the number of decimal places in the product. Study the given example.

EXAMPLE 15

Multiply.

a. 24.68×0.5

b. $(-213.6) \times (-2.5)$

Solutions:

a.

$$\begin{array}{r l l} 24.68 & \leftarrow & \text{2 decimal places} \\ \times \quad 0.5 & \leftarrow & \text{1 decimal place} \\ \hline 12.340 & \leftarrow & \text{3 decimal places} \end{array}$$

You may write the product as 12.34. Why?

b.

$$\begin{array}{r l l} -213.6 & \leftarrow & \text{1 decimal place} \\ \times \quad (-2.5) & \leftarrow & \text{1 decimal place} \\ \hline 534.00 & \leftarrow & \text{2 decimal places} \end{array}$$

The product is positive. Why?

Recall that when multiplying fractions, multiply the numerators to get the numerator of the product. Then multiply the denominators to get the denominator of the product.

$$\frac{a}{b} \bullet \frac{c}{d} = \frac{a \bullet c}{b \bullet d}$$

If the numerator and the denominator of the fractions that you need to multiply have common factors, you may cancel out the common factors first before multiplying the fractions.

When multiplying mixed numbers, write each mixed number as an improper fraction before multiplying.

EXAMPLE 16

Multiply. Write your answer in simplest form.

a. $-6\frac{6}{7} \bullet \frac{3}{8}$

b. $\left(-2\frac{5}{8}\right)\left(-3\frac{3}{7}\right)$

Solutions:

a.

$$\begin{aligned} -6\frac{6}{7} \bullet \frac{3}{8} &= -\frac{\overset{6}{\cancel{48}}}{7} \bullet \frac{3}{\cancel{8}} \\ &= -\frac{18}{7} \\ &= -2\frac{4}{7} \end{aligned}$$

Write $-6\frac{6}{7}$ as $-\frac{48}{7}$.

Since the factors have unlike signs, attach a negative sign to the product.

b. $\left(-2\frac{5}{8}\right)\left(-3\frac{3}{7}\right) = \left(-\frac{\overset{3}{\cancel{21}}}{\cancel{8}}\right)\left(-\frac{\overset{3}{\cancel{24}}}{\cancel{7}}\right)$ Write $-2\frac{5}{8}$ as $-\frac{21}{8}$ and $-3\frac{3}{7}$ as $-\frac{24}{7}$.

$= 9$ Cancel out the common factors.

EXAMPLE 17

Jam is a call center agent. She earns ₱85 per hour for the first 8 hours of work. She also earns an overtime pay of 1.5 times the hourly rate for each hour in excess of 8 hours. Suppose that Jam works for 10 hours each day from Monday to Wednesday. She also works for 11 hours each day on Thursday and Friday. How much is her salary for the given 5-day period?

Solution:

To find Jam's salary for the given period, compute for her salary for each specified day. It is given that for the first 8 hours, she is paid ₱85 per hour. Her overtime pay per hour is ₱85 • 1.5 = ₱127.50.

Day	Hours of Work	Pay for the First 8 Hours (in pesos)	Overtime Pay (in pesos)
Monday	10	85 • 8 = 680	127.50 • 2 = 255
Tuesday	10	85 • 8 = 680	127.50 • 2 = 255
Wednesday	10	85 • 8 = 680	127.50 • 2 = 255
Thursday	11	85 • 8 = 680	127.50 • 3 = 382.50
Friday	11	85 • 8 = 680	127.50 • 3 = 382.50
	Total	**3,400**	**1,530**

Therefore, Jam's total salary for the given period is ₱3,400 + ₱1,530 = ₱4,930.

Could you think of another way of solving the previous problem?

Division of Rational Numbers

Recall the following steps when dividing a number by a decimal:

Step 1. Make the divisor a whole number. You may do this by moving the decimal point of the divisor to the right until you have a whole number divisor.

Step 2. Move the decimal point of the dividend the same number of places to the right.

Step 3. Divide as you would with whole numbers. Note that the decimal point of the quotient is placed directly above the decimal point of the dividend.

Study the next example that shows how to solve a word problem involving division of decimals.

EXAMPLE 18

Lightning is a massive flow of electric current in the atmosphere. A bolt of lightning can reach temperatures approaching 30,000°C. It can contain one billion volts of electricity.

In electric devices, the amount of electric current can be computed using the formula $I = \frac{V}{R}$, where I is the electric current (in amperes), V is the voltage (in volts), and R is the resistance (in ohms). Compute the electric current flowing through a light bulb with a resistance of 10.5 ohms and that is plugged on a 110-volt outlet.

Solution:

To solve for the amount of electricity flowing through the light bulb, substitute the given values of V and R in the formula for I, then solve.

It is given that R = 10.5 ohms and V = 110 volts. So you will have:

$$I = \frac{110}{10.5}$$

To find I, divide 110 by 10.5.

```
         10.476...
105) 1100.000
    – 105
      50
    –  0
      50 0
    – 42 0
       8 00
    –  7 35
         650
    –    630
          20
```

Hence, the amount of electricity flowing through the light bulb is about 10.48 amperes.

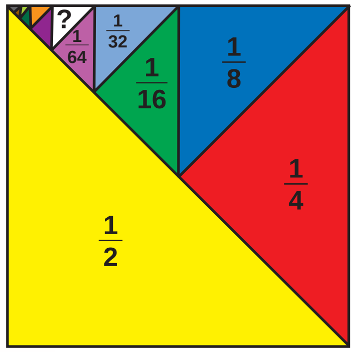
Recall that when dividing fractions, multiply the dividend by the reciprocal or multiplicative inverse of the divisor.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \bullet \frac{d}{c}.$$

Study the next example about solving a word problem involving division of fractions.

EXAMPLE 19

LJ, an interior designer, uses patterns to create quilts. One of her designs is shown on the right. What part of the square quilt is the white triangle?



Solution:

Note that the largest part in her quilt is $\frac{1}{2}$. The next largest part is obtained by dividing the remaining part of the triangle into 2; that is:

$$\frac{1}{2} \div 2 = \frac{1}{2} \bullet \frac{1}{2}$$
$$= \frac{1}{4}$$

The next fraction is also obtained in the same way; that is:

$$\frac{1}{4} \div 2 = \frac{1}{4} \bullet \frac{1}{2} = \frac{1}{8}$$

So, to solve for the required fraction, divide $\frac{1}{64}$ by 2.

$$\frac{1}{64} \div 2 = \frac{1}{64} \bullet \frac{1}{2} = \frac{1}{128}$$

Hence, the white triangle is $\frac{1}{128}$ of the square quilt.

Simplifying Complex Fractions

In some cases, a fraction may contain one or more fractions in its numerator and/or denominator. Such a fraction is called a *complex fraction*.

Complex fractions may involve a series or combination of operations on rational numbers. In such cases, the GEMDAS rule may be applied. Recall that GEMDAS stands for Grouping Symbols, Exponents, Multiplication, Division, Addition, and Subtraction. Study the next examples.

EXAMPLE 20

In 2003, Dagupan City in Pangasinan is the record holder of the world's longest barbecue. The barbecue measures about 3,300 ft. It was composed of milkfish placed together on grills arranged side by side. Suppose that a milkfish has an average width of $3\frac{3}{4}$ in. About how many pieces of milkfish were used for the barbecue?

Solution:

To find the number of milkfish, divide the total length of the barbecue by the width of each milkfish. Since the units are not the same, you may convert 3,300 ft. to inches. Since 1 ft. = 12 in., then 3,300 ft. = 39,600 in.

Now divide 39,600 by $3\frac{3}{4}$.

$$\frac{39{,}600}{3\frac{3}{4}} = \frac{39{,}600}{\frac{15}{4}}$$

$$= 39{,}600 \bullet \frac{4}{15}$$

$$= 10{,}560$$

Therefore, there were about 10,560 pieces of milkfish in the longest barbecue.

EXAMPLE 21

Simplify the following complex fraction: $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{2}{3}}$.

Solution:

One way to simplify the given expression is to express the numerator as a single fraction, and then to treat the fraction bar as a division symbol.

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{2}{3}} = \frac{\frac{2+3}{(2)(3)}}{\frac{2}{3}}$$ ← Add the fractions in the numerator.

$$= \frac{\frac{5}{6}}{\frac{2}{3}}$$ ← The fraction bar means ÷.

$$= \frac{5}{\cancel{6}_2} \cdot \frac{\cancel{3}^1}{2}$$ ← Multiply the numerator by the reciprocal of the divisor, and then simplify.

$$= \frac{5}{4}$$

Practice 2-3

Concepts and Skills

Give one rational number between each pair of rational numbers.

1. $1\frac{3}{8}$ and $1\frac{1}{4}$

2. $-\frac{3}{7}$ and -0.428

Plot each set of rational numbers on a number line. Then order the numbers from least to greatest.

3. $-1.9, 2.5, \frac{4}{3}, \frac{17}{9}$

4. $\frac{1}{4}, \frac{3}{7}, \frac{4}{19}, \frac{7}{20}$

5. $0.16, 0.16\overline{25}, 0.1\overline{63}, 0.\overline{16}$

Write each decimal as a fraction. Express your answer in simplest form.

6. 0.8

7. –6.75

8. 10.35

9. 5.66...

10. –0.3636...

Write each fraction as a decimal.

11. $\frac{5}{12}$

12. $-\frac{4}{9}$

13. $\frac{18}{7}$

14. $3\frac{1}{2}$

15. $-10\frac{3}{4}$

Write each fraction in simplest form.

16. $\frac{24}{54}$

17. $\frac{3}{51}$

18. $\frac{-72}{98}$

19. $-\frac{56}{108}$

20. $\frac{26}{143}$

Perform the indicated operations. Express your answer in simplest form.

21. $5\frac{1}{4}-2\frac{1}{3}$

22. $-9\frac{1}{3}+2\frac{1}{5}$

23. $30-2\frac{7}{10}$

24. $7\frac{3}{4}-\left(-2\frac{5}{7}\right)$

25. $\left(-3\frac{1}{5}\right)\left(4\frac{3}{8}\right)$

26. $(8)\left(-3\frac{5}{16}\right)$

27. $1\frac{3}{4}\div\frac{1}{2}$

28. $-12\frac{1}{4}\div 4\frac{1}{8}$

29. $\left(\frac{1}{3}+\frac{1}{2}\right)-\frac{5}{6}$

30. $9\div\frac{1}{3}-2\frac{1}{7}$

31. $\frac{1}{2}\bullet\left(16\frac{1}{4}-1\frac{2}{3}\right)$

32. $\dfrac{\frac{5}{12}}{\frac{1}{2}+\frac{1}{3}}$

33. $\dfrac{1}{1+\dfrac{1}{2+\frac{1}{2}}}$

34. $\dfrac{-\frac{1}{3}}{3-\frac{1}{9}}$

35. $(-5.5)(1.09 + 9.91)$

36. $\dfrac{(0.25)(-4.8)}{4.8-(-0.2)}$

37. $\dfrac{(0.\overline{3})(9)}{\frac{1}{3}+0.\overline{4}}$

Applications

Answer each item completely.

38. Mitch scores 45 out of 62 in a Math exam. She scores 36 out of 50 items in an English exam. In which subject is her score better?

39. Engineers set standard measurements for hand tools such as wrenches of different sizes. Engineer Joy has four wrenches with sizes $\frac{9}{16}$, $\frac{11}{16}$, $\frac{1}{2}$, and $\frac{5}{8}$ of an inch. Suppose that she needs the wrench with the second largest size. Which of the wrenches should she use?

40. A school year usually has 200 days. There are 365 days in 1 ordinary year. What fraction could represent a school year as part of an ordinary year? Express the fraction in simplest form.

41. The brain of a newborn baby weighs about 400 g. Suppose that a newborn baby weighs 3.5 kg. What fraction of the baby's weight is his or her brain?

42. A hair strand is approximately $\frac{1}{450}$ in. thick. What is the equivalent value of this fraction in decimal form?

43. Jose has 4 pieces of wire. Two of the pieces of wire measure 9 in. long each. The other 2 pieces measure $1\frac{1}{4}$ ft. each. If he puts the pieces of wire together, how long is the wire in inches?

44. Dennis has finished painting $\frac{1}{3}$ of his room using $2\frac{1}{2}$ cans of paint. How many more cans of paint will he need to finish painting his room?

45. The fuel tank of Trina's car is $\frac{2}{3}$ full. To travel to her destination, she needs a $\frac{4}{5}$-full tank. What part of the fuel tank does she need to fill to reach her destination?

46. There are 18 baskets to be filled with different kinds of fruits. Suppose that there are $1\frac{1}{2}$ dozen pineapples, 6 dozen apples, 3 dozen oranges, $7\frac{1}{2}$ dozen bananas, and 9 dozen mangoes. If each basket should have the same contents, how many pieces of each fruit must be placed in each basket?

47. Mr. Angeles shared $\frac{1}{2}$ of his estate with his wife. Also, he shared $\frac{1}{8}$ with each of his two children, and $\frac{1}{16}$ with each of his three grandchildren. He donated the remaining ₱30,000 worth of property to a charity. What is the value of his entire estate?

48. A carpenter will cut a $42\frac{3}{4}$-in. long board into 5 pieces of equal lengths. If $\frac{1}{16}$ in. of board wears down in every cut, how long will each piece be?

49. Suppose that 0.408 kg of grapes costs ₱65.28. How much is 1 kg of grapes?

50. The following table shows the monthly average exchange rate of US dollar to peso in the first six months of the year 2016:

Month	Peso Equivalent of 1 US dollar
January	47.73
February	47.45
March	46.64
April	46.20
May	46.67
June	43.32

What is the average exchange rate for the first six months of that year?

Enrichment Exercises

Answer each item completely.

51. Can you find unique integers *a, b,* and *c* such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$? If yes, give a possible set of integers. If no, explain why.

52. Look at the following method of renaming fractions:

$$\frac{1\not{6}}{\not{6}4} = \frac{1}{4} \qquad \frac{1\not{9}}{\not{9}5} = \frac{1}{5} \qquad \frac{4\not{9}}{\not{9}8} = \frac{4}{8}$$

Can you find other fractions for which this method works? Also, give an example for which this method does not work and explain why.

53. Find the sum of the following expression:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{99\cdot 100}.$$

54. Consider the following equation:

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}=\frac{10}{14}=\frac{5}{7}$$

Is the equation correct? Explain.

55. The following figure is called a magic square:

$6\frac{1}{4}$		$11\frac{1}{4}$
	$7\frac{1}{2}$	
		$8\frac{3}{4}$

Complete the magic square. Each row, column, and diagonal must add up to the same number.

LESSON 2-4

Irrational Numbers

Learning Objectives

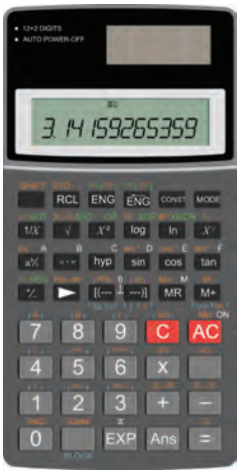
At the end of the lesson, you should be able to:

- define irrational numbers
- locate an irrational number on a number line
- find the square roots of positive rational numbers

PROBE AND LEARN

Rainbows are among the most beautiful natural phenomena. A rainbow is formed when light shines through water droplets, usually in the form of rain, and diffracts into different colors. A rainbow is not a 360° full circle; it is just a 42° arc around a spot opposite the Sun.

To explain natural phenomena that exhibit circles or spheres, the number π (pi) is usually used. This number is an example of an *irrational number*.



Using a scientific calculator, you can get the value of π by pressing the π button. A 12-digit scientific calculator will display 3.14159265359, while a 15-digit calculator will show 3.14159265358979. Notice that when you use a calculator that can display more digits, the number of displayed digits of π also increases. This indicates that π is nonterminating. Recall that nonterminating but repeating decimals are classified as rational numbers. Take note, however, that nonterminating decimals, such as π, that do not show a repeating pattern of digits are *irrational numbers*.

Set of Irrational Numbers

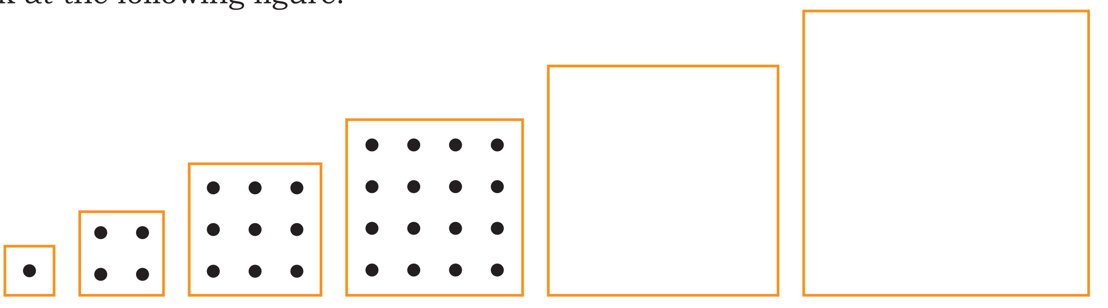
Key Concept

The set of **irrational numbers** is the set of numbers that cannot be expressed as $\frac{p}{q}$, where p and q are integers, and $q \neq 0$.

To better understand the set of irrational numbers, you need to learn how to find the square roots of positive rational numbers.

Square Roots of Positive Rational Numbers

Interesting patterns may be discovered by representing numbers with figures. Look at the following figure:



Observe how dots were drawn inside each square. Do you notice a pattern? Draw dots in the next two squares based on the pattern. Then complete the following table:

Number of Dots along Each Side of the Square	Total Number of Dots
1	1
2	4
3	9
4	16
5	
6	
7	
8	
9	
10	

Do you know?

Some numbers are regarded as *happy numbers*. Consider the number 13. Adding the squares of the digits of 13 gives $1^2 + 3^2 = 10$. Repeating the process with the digits of 10 yields $1^2 + 0^2 = 1$. Since the resulting number is 1, then 13 is a happy number. For any happy number, the resulting number should be 1 after a finite number of steps.

How many happy numbers less than 50 can you find?

The total number of dots in each square represents a *perfect square*. The number of dots in each row or column in the square is called a *square root* of the perfect square. For example, 16 is a perfect square and 4 is its square root.

Key Concept

If a and b are nonnegative numbers and $b^2 = a$, then b is a **square root** of a. In symbols, it can be written as:

$$\sqrt{a} = b.$$

Consider $\sqrt{16} = 4$, which is true since $4^2 = 4 \bullet 4 = 16$. The number 16 is called the *radicand*. The number 4 is the square root. However, note that $(-4)(-4)$ is also equal to 16. Does this mean that -4 is also a square root of 16?

The expression $\sqrt{x}$ is used to refer specifically to the *principal square root of x*, or the nonnegative square root of x. While it is true that $(-4)(-4) = 16$, -4 is not the principal square root of 16, but the *negative square root* of 16.

$\sqrt{16} = 4$ → 4 is the principal square root of 16.

$-\sqrt{16} = -4$ → -4 is the negative square root of 16.

Perfect squares also include decimals and fractions. An example is 0.04 since $0.04 = (0.2)^2$. Another example is $\frac{9}{16}$ since $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$.

Example 1

Find each indicated square root.

a. $\sqrt{225}$

b. $-\sqrt{49}$

Solutions:

a. $\sqrt{225} = 15$ — $15^2 = 225$

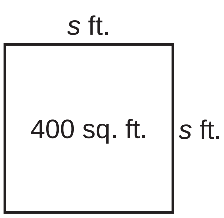
b. $-\sqrt{49} = -7$ — $-\sqrt{\ }$ indicates the negative square root.

Example 2

Four hundred pieces of 1-square-foot tiles are used for the floor tiling of a square room. What is the measure of each side of the room?

Solution:

It is given that 400 pieces of 1-square-foot tiles are used for floor tiling. Thus, the area of the room is 400 square feet (sq. ft.). To find the measure of each side of the square room, use the formula $A = s^2$, where A is the area and s is the measure of each side of the square. But since you need to find s, you must get the square root of A; that is, $s = \sqrt{A}$.

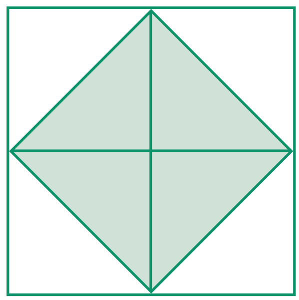


Replacing A with 400 in the given formula gives $s = \sqrt{400}$.

Since $400 = 20^2$ and $400 = (-20)^2$, then s has two possible values: 20 or –20. However, since s represents the measure of each side of the room, it cannot be negative. Therefore, each side of the room measures 20 ft.

You have seen that extracting the square root of a perfect square yields a rational number. Consider another situation.

A dressmaker designs a cover for square tables that measure 2 ft. on each side. She drafts the design shown on the right.



Since the square table measures 2 ft. on each side, the amount of cloth that she needs for each cover is 4 sq. ft. Connecting the midpoints of the sides of the square forms a small square, which is the shaded part. Such small square has an area of 2 sq. ft. What could be the measure of each side of the smaller square?

Since the area of the small square is 2 sq. ft., the measure of its side must be a number s which, when squared, gives 2; that is, $s^2 = 2$. What is the value of s?

You may try to guess and check possible values of s. For example, you know that $1^2 = 1$ and $2^2 = 4$. Since $s^2 = 2$, and 2 is between 1 (or 1^2) and 4 (or 2^2), you can say that s is between 1 and 2. Note, however, that no integer lies between 1 and 2. This implies that s is not an integer.

Recall that a rational number can be written as a ratio of two integers, where the denominator is not 0. The equation $s^2 = 2$ is equivalent to $s = \frac{2}{s}$. If $\frac{2}{s}$ is a rational number, then the denominator s has to be an integer. However, you already know that no such integer exists. This means that s is neither an integer nor a rational number.

Problems like this involve irrational numbers. The solution to this problem is the irrational number $\sqrt{2}$. It is equivalent to a nonterminating and nonrepeating decimal. The value of $\sqrt{2}$, correct to 10 decimal places, is 1.4142135623....

Key Concept

In general, the square root of a non-perfect square is an irrational number.

For example, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, and $\sqrt{10}$ are irrational numbers. You can, however, express the square roots of non-perfect square numbers, correct to a specific number of decimal places. Study the next example.

EXAMPLE 3

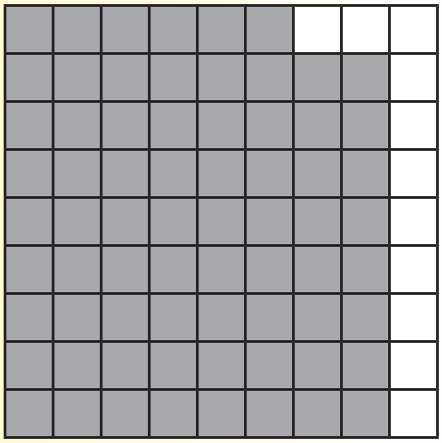
Find a whole number approximation for $\sqrt{70}$.

Solution:

Since $8^2 = 64$ and $9^2 = 81$, you can say that 70 is between the two perfect squares, 64 and 81. Hence, $\sqrt{70}$ is between 8 and 9.

$$64 < 70 < 81$$
$$8^2 < 70 < 9^2$$
$$8 < \sqrt{70} < 9$$

Between 8 and 9, which is a closer whole number approximation of $\sqrt{70}$? In the following 9 × 9 grid, you can see that the 70 shaded squares represent a number that is closer to 64 squares than to 81 squares.



Thus, the whole number approximation for $\sqrt{70}$ is 8.

For a close approximation of a square root up to a specific number of decimal places, you may follow the steps shown in the next example.

EXAMPLE 4

Approximate $\sqrt{8}$ up to two decimal places.

Solution:

Step 1. Determine between which two perfect squares the radicand lies. Then get the square roots of such perfect squares.

The radicand is 8, and 8 is between the perfect squares 4 and 9.

$$4 < 8 < 9$$
$$2^2 < 8 < 3^2$$
$$2 < \sqrt{8} < 3$$

Therefore, $\sqrt{8}$ is between 2 and 3.

Step 2. Divide the radicand by any of the square roots you obtained in step 1. Suppose that you divide 8 by 3.

$$8 \div 3 \approx 2.67$$

(The quotient may be rounded to the nearest hundredths.)

Step 3. Find the average of the quotient in step 2 and the square root you used as a divisor.

So get the average of 2.67 (which is the quotient) and 3 (which is the divisor).

$$\frac{2.67+3}{2} = \frac{5.67}{2}$$

$$= 2.835$$

Step 4. Check if the value you obtained in step 3 is a good estimation of the required square root. To do this, get the square of the obtained value. Compare it with the given radicand by getting the absolute value of the difference of the result and the radicand.

$$(2.835)^2 \approx 8.0372$$

The difference between this result and the radicand is:

$$|8.0372 - 8| = 0.0372.$$

You can check if you can still find a closer estimation. Do the next step.

Step 5. Repeat steps 2 and 3, but this time, use the value you obtained in step 3 as the divisor.

So divide 8 by 2.835.

$$8 \div 2.835 \approx 2.822$$

Get the average of 2.835 and 2.822.

$$\frac{2.835+2.822}{2} = 2.829$$

Step 6. Repeat step 4. Then compare the two estimates you obtained. Choose the estimate whose square is closer to the radicand. To do this, get the absolute value of the difference of each square and the radicand.

$$(2.829)^2 = 8.0032$$

Between 8.0372 and 8.0032, the number closer to 8 is 8.0032 since:

$$|8.0032 - 8| = 0.0032 \quad \text{and} \quad |8.0372 - 8| = 0.0372.$$

Thus, you can say that 8 is approximately equal to 2.829. When you compute $\sqrt{8}$ using a calculator, you will get 2.828427125. As you can see, you obtained a close approximation for $\sqrt{8}$.

Knowing how to approximate square roots of non-perfect squares can help you plot irrational numbers on a number line. From the previous example, since $\sqrt{8}$ is approximately equal to 2.829, you can plot it on the number line as follows:

EXAMPLE 5

One of Manila's major attractions is the breathtaking sunset view along the Baywalk at Manila Bay. The line that seems to separate the bay and the sky is called the *horizon*. How far you can see to the horizon can be determined using the formula $d = 1.22\sqrt{h}$, where d is the distance you can see (in miles), and h is the height (in feet) your eyes are from the ground. Suppose that your eyes are about 5 ft. from the ground. Estimate how far you can see to the horizon during a sunset if you are standing along the Baywalk.

Solution:

Use the formula $d = 1.22\sqrt{h}$, where h = 5.

$$\begin{aligned} d &= 1.22\sqrt{h} \\ &= 1.22(\sqrt{5}) \\ &\approx 1.22(2.24) \\ &\approx 2.7328 \end{aligned}$$

Thus, you can see to the horizon up to a distance of about 2.7 miles.

PRACTICE 2-4

Concepts and Skills

Without using a calculator, determine whether each rational number is a perfect square or not.

1. 16
2. 20
3. 121
4. 225
5. 1,000
6. $\frac{1}{81}$
7. $-\frac{49}{100}$
8. 0.0004
9. 0.0404
10. 4.0004

Find each square root without using a calculator.

11. $\sqrt{36}$
12. $\sqrt{64}$
13. $-\sqrt{121}$
14. $-\sqrt{225}$
15. $\sqrt{400}$
16. $\sqrt{\frac{4}{9}}$
17. $\sqrt{0.16}$
18. $\sqrt{\frac{144}{625}}$
19. $-\sqrt{1.96}$
20. $-\sqrt{0.0009}$

Without using a calculator, find the nearest whole number approximation for each irrational number.

21. $\sqrt{5}$
22. $\sqrt{50}$
23. $\sqrt{121}$
24. $\sqrt{27.5}$
25. $\sqrt{\frac{268}{7}}$

Without using a calculator, find an approximation for each square root up to the indicated place value.

26. $\sqrt{3}$ to the nearest tenths
27. $\sqrt{10}$ to the nearest hundredths
28. $\sqrt{27.5}$ to the nearest tenths

29. $\sqrt{40}$ to the nearest tenths

30. $\sqrt{101}$ to the nearest hundredths

Answer each question. Choose the letter of your answer.

31. Which statement is *false*?

A. $\sqrt{0} = 0$

B. $\sqrt{1} = -1$

C. $\sqrt{1} = 1$

D. $-\sqrt{1} = -1$

32. What is $-\sqrt{196}$?

A. –14 B. –13 C. 13 D. 14

33. Between which two integers does $\sqrt{220}$ lie?

A. 13 and 14

B. 14 and 15

C. 16 and 17

D. 17 and 18

34. Which number must be a perfect square?

A. sum of two perfect squares

B. product of two odd numbers

C. product of two perfect squares

D. product of an even number and a perfect square

35. Which number has a square root that is *not* a real number?

A. a counting number

B. a negative number

C. a positive irrational number

D. zero

For items **36** to **40**, find the value of each expression.

36. $\sqrt{1{,}156}$

A. 24 B. 34 C. 36 D. 44

37. $\sqrt{400.8004}$

A. 2.02 B. 2.2 C. 20.02 D. 20.2

38. $\sqrt{0.64} + \sqrt{0.0064}$

A. 0.08 B. 0.088 C. 0.8 D. 0.88

39. $\sqrt{16+9}$

A. 3 B. 4 C. 5 D. 7

40. $\sqrt{\sqrt{625}}$

A. 5 B. 15 C. 25 D. 125

Find the next two numbers in each sequence.

41. $1, \sqrt{2}, 2, 2\sqrt{2}, 4, 4\sqrt{2}, 8, 8\sqrt{2}$, ______, ______

42. $1, 2\sqrt{2}, 3\sqrt{3}, 8, 5\sqrt{5}, 6\sqrt{6}, 7\sqrt{7}$, ______, ______

43. –1, 1, –9, 9, –25, 25, –49, 49, ______, ______

44. $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}$, ______, ______

45. –1, 0, 0, 1, 1, 2, 4, 5, 25, 26, ______, ______

Applications

Answer each item completely.

46. The Great Pyramid of Khufu is known as the largest pyramid ever built in Egypt. Its base forms a nearly perfect square with an area of 52,900 m^2. What is the measure of each side of its base?

47. Jon is reading a certain page of a book. The square of the page is 961. On which pages is Jon's book open?

48. A farmer grows cabbages in his square-shaped farm. Each cabbage takes 1 sq. ft. of land area in his farm. He planted 256 cabbages. How many cabbages are planted along each side of his farm?

49. A square pigpen has an area of 576 sq. ft. How many feet of fencing material is required to enclose it?

50. A man is at the top of a mountain and his eyes are approximately 500 ft. above the ground. How far can he see to the horizon? Use the formula in example 5 of this lesson. Express your answer to the nearest whole number.

Enrichment Exercises

Answer each item completely.

51. Find two integers whose square is 225.

52. Suppose that a perfect square is divided by 4. What are the possible remainders?

53. Write an expression that represents the relationship between a perfect square and its square root.

54. Name all whole numbers whose square roots are between 7 and 8.

55. The ones digit of a perfect square is the same as the ones digit of its square root. What are the possible ones digits of the number?

Chapter Assessment

Answer each question without using a calculator. Choose your answer.

1. What is the simplest form of $\frac{28}{91}$?

A. $\frac{4}{17}$ B. $\frac{4}{13}$ C. $\frac{7}{17}$ D. $\frac{7}{13}$

2. Which is equivalent to $0.\overline{36}$?

A. $\frac{1}{36}$ B. $\frac{9}{25}$ C. $\frac{4}{11}$ D. $\frac{9}{11}$

3. What is the LCD of $\frac{9}{28}$ and $\frac{13}{91}$?

A. 4 B. 7 C. 28 D. 364

4. Which is a rational number?

A. $\sqrt{24}$
B. 4.56565653...
C. 0.123456...
D. 5.3613613613...

5. Which is an irrational number?

A. 3.123123...
B. 3.14
C. $\sqrt{25}$
D. $\sqrt{50}$

6. Which set of fractions is arranged from greatest to least?

A. $\frac{9}{8}, \frac{7}{12}, \frac{3}{7}, \frac{6}{13}$
B. $\frac{9}{8}, \frac{7}{12}, \frac{6}{13}, \frac{3}{7}$
C. $\frac{3}{7}, \frac{6}{13}, \frac{7}{12}, \frac{9}{8}$
D. $\frac{6}{13}, \frac{7}{12}, \frac{3}{7}, \frac{9}{8}$

7. Which set is *not* a subset of the set of rational numbers?

A. set of terminating decimals
B. set of integers
C. set of non-terminating decimals
D. set of whole numbers

8. Which set is closed under division?

A. set of counting numbers
B. set of integers
C. set of non-zero rational numbers
D. set of whole numbers

9. What is the value of $\sqrt{784}$?

A. 22 B. 28 C. 32 D. 38

10. Which decimal is greater than 0.25 but less than $\frac{2}{7}$?

A. 0.249 B. $0.\overline{28}$ C. $0.2\overline{8}$ D. 0.29

11. A fruit dealer delivers 4 crates of mangoes. He unloads the heaviest crate from his van last. Which crate does he unload last?

A. crate A weighing 27.875 kg
B. crate B weighing 28.105 kg
C. crate C weighing 27.984 kg
D. crate D weighing 28.052 kg

12. Larry withdraws ₱1,000 from his bank account. Which expression represents the amount in his transaction?

A. –1,000 B. 1,000 C. $|-1{,}000|$ D. $|1{,}000|$

13. A recipe that makes 12 servings requires $\frac{3}{4}$ cup of melted butter. Suppose that you will prepare a recipe that makes only 9 servings. How many cups of melted butter do you need?

A. $\frac{1}{4}$ cup B. $\frac{3}{8}$ cup C. $\frac{1}{2}$ cup D. $\frac{9}{16}$ cup

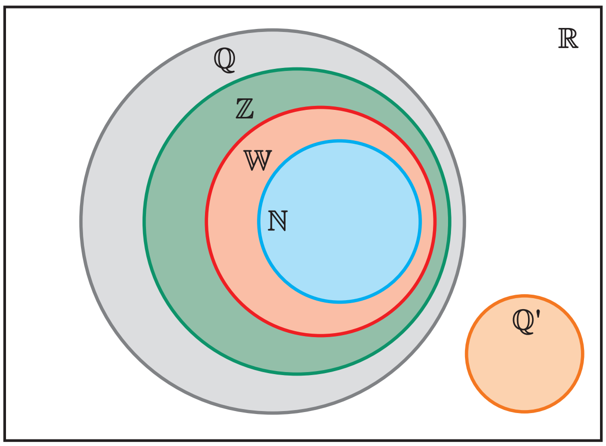
14. On a particular day, the exchange rate of euro (€) to Philippine peso (₱) is €1 = ₱65.825. The next day, the peso equivalent of €1 is 15 centavos less. How much is €10 based on the last exchange rate?

A. ₱656.25 B. ₱656.75 C. ₱658.10 D. ₱658.25

15. A one-hectare piece of land is square in shape. What is the length of each side of the piece of land?

A. 1 m B. 10 m C. 100 m D. 1,000 m

Locate each number in its correct place in the Venn diagram on the right.



16. $\sqrt{20}$

17. –20

18. 2.22…

Write the property of real numbers that each equation illustrates.

19. $105 \bullet 624 = (100 + 5) \bullet 624$

20. $(40 + 8) \bullet 25 = 25 \bullet (40 + 8)$

21. $\frac{2}{3} \bullet \frac{3}{2} = 1$

Compare each pair of values using >, <, or =.

22. $|-24|$ ______ $|24|$

23. $\frac{1}{2}$ ______ $\sqrt{0.25}$

24. $-\sqrt{49}$ ______ $|-7|$

25. $-\frac{5}{7}$ ______ $\frac{7}{-9}$

26. 0 ______ -102

27. $|-21|$ ______ $|-32|$

28. -0.3845 ______ -0.3467

29. $\left|-\frac{3}{11}\right|$ ______ 0.27

Perform the indicated operations. Express your answer in simplest form.

30. $\frac{-2.5 - 3}{-0.5}$

31. $(-36) - (-28) + 42$

32. $\frac{\frac{1}{3}}{\frac{1}{2}} + \frac{\frac{1}{4}}{\frac{1}{3}}$

33. $(-240)\left(\frac{11}{5}\right)$

For items **34** to **37**, refer to the following table that shows the times recorded by a 100-m dash sprinter over a one-week training.

Day	Time (in seconds)
Monday	10.43
Tuesday	10.39
Wednesday	10.56
Thursday	10.45
Friday	10.46

34. On which day did the sprinter run fastest?

35. On which day did the sprinter run slowest?

36. How many seconds did the sprinter's time increase from Monday to Wednesday?

37. What is the average time of the sprinter over the one-week period?

For items **38** and **39**, consider the following situation:

A basketball team won 7 out of 12 games.

38. What fraction of the games did the team lose?

39. Write this fraction as a decimal.

For items **40** and **41**, refer to the following situation:

A school purchases 1,000 new chairs for its auditorium. The chairs are arranged in rows to form a square.

40. What is the maximum number of chairs that can be arranged in the auditorium?

41. How many chairs are in each row?

For items **42** and **43**, consider the following situation:

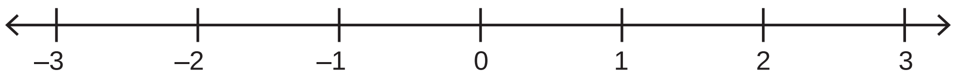
Randy receives a monthly salary of ₱12,000. He allots $\frac{1}{6}$ of his salary for his house rental and saves another $\frac{1}{6}$. He allots $\frac{3}{4}$ of the remaining amount for transportation and food. He donates the rest to a charity.

42. What fraction of his salary does he donate to a charity?

43. How much does he allot for transportation and food?

Answer each item completely.

44. Graph the points $\frac{1}{4}$, π, -2, $-\sqrt{5}$, and $6.\overline{6}$ on a number line.



45. On a certain day, the temperature dropped 18° in the Fahrenheit scale over a three-hour period. Suppose that the temperature dropped at a constant rate. How many degrees did the temperature fall each hour?

46. Karin has ₱23.75 left after buying a notebook and a ball pen. The notebook costs ₱37.50. The ball pen costs half as much as the notebook. How much did she have before buying the two items?

47. A large handkerchief requires 144 square inches of cloth. A small handkerchief requires 64 square inches. How much longer is the side of each cloth required for a large handkerchief than for a small handkerchief?

48. Danny and Jenny, plan to meet, but they have different work schedules. Danny is free only every 6 days. Jenny is off duty every 4 days. If they are both free on Wednesday this week, when is the next time they could meet?

49. Suppose that you are on an airplane flying through a clear sky at an altitude of approximately 32,400 ft. How far can you see to the horizon? Use the formula that you used in example 5 of Lesson 2-4.

50. Draw a Venn diagram that shows the set of real numbers and its subsets. Write at least two sample elements for each subset.